HW 7 Solutions

Sec. 31:

36. For all $n \in \mathbb{Z}, n \geq 2$ the polynomial $x^n - 2$ is irr. in $\mathbb{Q}[x]$ by the Eisenstein criterion with $P=2$. Now by contradiction suppose $\overline{\mathbb{Q}}$ has degree $r<\infty$. Then ther would be no alg. ext. of $\mathbb{Q}$ of deg. greater than r. But if α is a root of $x^{r+1}-2$ then $(\mathbb{Q}(\alpha):\mathbb{Q}) = r+1$, Contradiction.

37. Because $[\mathbb{C}:\mathbb{R}]=2$ and $\mathbb{C}$ is an alg. closure of $\mathbb{R}$, it must be that every irr. $f(x)$ in $\mathbb{R}[x]$ of deg. >1 is of deg. 2.

For any root $\alpha \in \mathbb{C}$ of such a poly. $f(x)$ we have:

$$2=[\mathbb{C}:\mathbb{R}] \geq [\mathbb{R}(\alpha):\mathbb{R}] = \deg f > 1 \Rightarrow \deg f = 2 \Rightarrow \mathbb{C} = \mathbb{R}(\alpha) \cong \mathbb{R}[x]/\langle f(x)\rangle.$$

Now let E any finite ext. of $\mathbb{R}$. If $E \neq \mathbb{R}$ then let $\beta \in E$ but not in $\mathbb{R}$. Then $f(x) = \operatorname{irr}(\beta, \mathbb{R})$ has deg. 2 because we have seen that there are no irr. poly. in $\mathbb{R}[x]$ of greater degree. From what was said above $\mathbb{R}(\beta) \cong \mathbb{R}[x]/\langle f(x)\rangle \cong \mathbb{C}$.

Since $\mathbb{C}$ is alg. closed $\Rightarrow \mathbb{R}(\beta)$ is alg. closed & hence has no alg. ext. $\Rightarrow \mathbb{R}(\beta)=E \Rightarrow E \cong \mathbb{C}$.

Sec. 32:

1. $e^{i(3\theta)} = \cos(3\theta) + i\sin(3\theta)$. On the other hand, $e^{i3\theta} =$
$= (e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3$ which by binomial thm equals:

$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta - 3\cos\theta\sin^2\theta + i(\cdots).$$

Thus $\cos(3\theta) = \cos^3\theta - 3\cos\theta(1-\cos^2\theta) = 4\cos^3\theta - 3\cos\theta$.

3. If a regular 9-gon could be constructed then the angle $360^\circ/9 = 40^\circ$ could be constructed & then could be bisected to construct an angle of 20°. However in the proof of Thm. 32.11 we showed that 20° is not constructible.