Last Name: 	Student Number: 
First Name: 

TIME ALLOWED: 75 MINUTES. TOTAL MARKS: 100
NO AIDS ALLOWED. WRITE SOLUTIONS ON THE SPACE PROVIDED.
PLEASE READ THROUGH THE ENTIRE TEST BEFORE STARTING
AND TAKE NOTE OF HOW MANY POINTS EACH QUESTION IS WORTH.
FOR FULL MARK YOU MUST PRESENT YOUR SOLUTION CLEARLY.

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1. 

(a) [10 points] Let \( f : S \to \mathbb{R} \) be a function. Give the following definition: 
\( f \) is **continuous** at a point \( c \in S \).

\[
f \text{ is contin. at } c \text{ if:} \\
\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } \forall x \in S \text{ whenever} \\
| x - c | < \delta \ \Rightarrow \ | f(x) - f(c) | < \varepsilon.
\]

(b) [10 points] State the **Intermediate Value Theorem**.

Let \( f : [a, b] \to \mathbb{R} \) be a contin. function.

Let \( f(a) < y < f(b) \) or \( f(b) < y < f(a) \). Then \( \exists c \in (a, b) \text{ such that } f(c) = y. \)
2. 

(a) [10 points] Prove continuity of composition of functions. More precisely, let \( A, B \subset \mathbb{R} \) and \( f : B \to \mathbb{R} \) and \( g : A \to \mathbb{R} \). Suppose \( g \) is continuous at \( c \in A \) and \( f \) is continuous at \( g(c) \). Prove that \( f \circ g : A \to \mathbb{R} \) is continuous at \( c \).

Let \( \{x_n\} \) be a seq. in \( A \) with \( \lim_{n \to \infty} x_n = c \). Then \( \lim_{n \to \infty} g(x_n) = g(c) \) by continuity of \( g \) (sequential limit thm., Prop. 3.2.2 in the text).

Applying the same thm. to the continuous function \( f \) & the seq. \( \{g(x_n)\} \), we conclude \( \lim_{n \to \infty} f(g(x_n)) = f(g(c)) \). Again using Prop. 3.2.2 (iii) \( \wedge \)

we see that \( f \circ g \) is contin. at \( c \).

(b) [10 points] Give a precise proof that the function \( f(x) = x^{1/2} \) is uniformly continuous on the interval \((1, \infty)\). (Hint: multiply with the conjugate.)

Let \( \varepsilon > 0 \) be given. we need \( \delta > 0 \) such that \( \forall x, y \in (1, \infty) \)

if \( |x-y| < \delta \) \( \Rightarrow \) \( \frac{|x^{1/2} - y^{1/2}|}{|x^{1/2} + y^{1/2}|} < \varepsilon \).

\( |x^{1/2} - y^{1/2}| < \varepsilon \) \( \iff \) \( \frac{|x-y|}{|x^{1/2} + y^{1/2}|} < \varepsilon \) \( \iff \) \( |x-y| < \varepsilon |x^{1/2} + y^{1/2}| = \varepsilon (\sqrt{x} + \sqrt{y}) \)

Now \( 1 < x, y \Rightarrow 2\sqrt{x+y} \Rightarrow \varepsilon (\sqrt{x} + \sqrt{y}) > 2\varepsilon \).

Take \( \delta = 2\varepsilon \).

\( |x-y| < 2\varepsilon \) \( \Rightarrow |x-y| < \varepsilon (\sqrt{x} + \sqrt{y}) \) \( \Rightarrow |x^{1/2} - y^{1/2}| = \frac{|x-y|}{\sqrt{x} + \sqrt{y}} < \varepsilon \) as required.
3.

(a) [10 points] State the **Bolzano-Weierstrass theorem** about sequences of real numbers.

Any bounded seq. of real numbers has a convergent subsequence.

(b) [10 points] Show that the series

$$\sum_{n=0}^{\infty} \frac{1}{1 + n + n^2}$$

is convergent. (You don’t need to find the limit.)

By p-series test, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent $\Rightarrow$ By comparison test, $\sum_{n=1}^{\infty} \frac{1}{1+n+n^2}$ is convergent $\Rightarrow$ Convergent $\Rightarrow \sum_{n=0}^{\infty} \frac{1}{1+n+n^2}$ also convergent.
4. [20 points] True or False? Justify your answers. You can use any material proved in class.

(a) There exists a function $f$ which is continuous at every point in $(0, 1)$ but is not uniformly continuous on $(0, 1)$.

$$f(x) = \frac{1}{x}. \quad f(x) \text{ is continuous (See Example 3.2.3).}$$

Let us show $f$ is not uniformly continuous.

If $f(x)$ was uniformly continuous by Thm.34.6, it would have a continuous extension to $[0,1]$. But it doesn’t since $\lim_{x \to 0} \frac{1}{x}$ D.N.E.

(b) There exists a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous at irrational numbers and is discontinuous at rational numbers.

Popcorn function.

$$f(x) = \begin{cases} 0 & x \text{ irrational} \\ \frac{1}{n} & x = \frac{m}{n} \text{ rational} \quad \gcd(n,m)=1 \end{cases}$$

(c) $\sqrt{2}$ is a cluster point of $\mathbb{Q}$, the set of rational numbers.

Every interval contains some rational number, thus

$\forall \epsilon > 0$ the interval $(\sqrt{2} - \epsilon, \sqrt{2} + \epsilon)$ intersects $\mathbb{Q} \Rightarrow \sqrt{2}$ is a cluster point of $\mathbb{Q}$. 
5.

(a) [10 points] Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function. Suppose that for a point \( x_0 \in \mathbb{R} \) the sequence:

\[
x_0, \ f(x_0), \ f(f(x_0)), \ f(f(f(x_0))), \ldots
\]

is convergent to a number \( a \in \mathbb{R} \). Prove that \( f(a) = a \). Clearly point out the statements/theorems you use in your proof. (Hint: use sequential limit property of continuous functions.)

Let \( x_n = \underbrace{f(\ldots(f(x_0))\ldots)}_{\text{n times}} \).

By assumption \( \lim_{n \to \infty} x_n = a \). Since \( f \) is assumed to be continuous \( \Rightarrow \)

\[
\lim_{n \to \infty} f(x_n) = f(a).
\]

But \( f(x_n) = x_{n+1} \Rightarrow \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_{n+1} = a \)

(\( \text{Tail of a seq. has the same limit.} \))

Since a seq. has unique limit we conclude that \( a = f(a) \).

(b) [10 points] Consider the sequence defined inductively by \( x_1 = 1 \), and

\[
x_{n+1} = \frac{1}{1 + x_n}
\]

for \( n \geq 1 \). Suppose we know that the sequence \( \{x_n\} \) is convergent. Use (a) to find its limit.

Aside: One can think of this limit as the continued fraction:

\[
a = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}}.
\]

Let \( a = \lim_{n \to \infty} x_n \).

By (a) we must have \( a = \frac{1}{1 + a} \).

\[
\Rightarrow a^2 + a - 1 = 0 \Rightarrow a = \frac{-1 \pm \sqrt{5}}{2} > 0 \text{ or } \frac{-1 - \sqrt{5}}{2} < 0.
\]

Note that if \( x > 0 \Rightarrow \frac{1}{1 + x} > 0 \). By induction it follows that \( \forall n \in \mathbb{N} \)

\[
x_n > 0 \Rightarrow \lim_{n \to \infty} x_n = a > 0. \text{ So } a = \frac{-1 + \sqrt{5}}{2} \text{ (because the other option is negative).}
\]

(The number \( \frac{-1 + \sqrt{5}}{2} \) is the golden ratio.)
6. [1 point] Draw a cartoon of a mathematician!