Solutions

**Department of Mathematics**

## University of Pittsburgh

## MATH 1050 (Combinatorics)

## Midterm 1 (Fall 2015)

**Last Name:** **Student Number:**

**First Name:**

TIME ALLOWED: 50 MINUTES. TOTAL: 50+2
NO AIDS ALLOWED. WRITE SOLUTIONS ON THE SPACE PROVIDED.

| Question | Mark |
|---|---|
| 1 | /10 |
| 2 | /10 |
| 3 | /10 |
| 4 | /10 |
| 5 | /10 |
| 6 | /2 |
| TOTAL | /50 + 2 |

**1(a).**[7 points] Give the definition of the following: (i) A tree (graph theory), (ii) A Hamiltonian graph (graph theory).

Tree: A connected graph without any cycles.

Hamiltonian graph: A graph that has a cycle which passes through all the vertices.

**(b).**[3 points] State the Pigeon Hole principle.

Let $X$ & $Y$ be finite sets. Let $f: X \longrightarrow Y$ be any function.

If $|X| > |Y|$ then $\exists x_1, x_2 \in X$ such that $x_1 \neq x_2$ & $f(x_1) = f(x_2)$.

**2.**[10 points] How many lattice paths (in 3-dimensional space) are there from $(0,0,0)$ to $(5,5,5)$? (There are 3 moves allowed in the lattice path: adding $+1$ to $x$-coordinate, adding $+1$ to $y$-ccordinate and adding $+1$ to $z$-coordinate.)

$\binom{15}{5,5,5} = \frac{15!}{5!\,5!\,5!}$ multinomial number

Let W, E, U denote west, east and up moves. Each lattice path from $(0,0,0)$ to $(5,5,5)$ corresponds to a seq.^n of length 15 consisting of 5 W's, 5 E's & 5 U's. Number of such sequences is the multinomial number $\binom{15}{5,5,5}$.

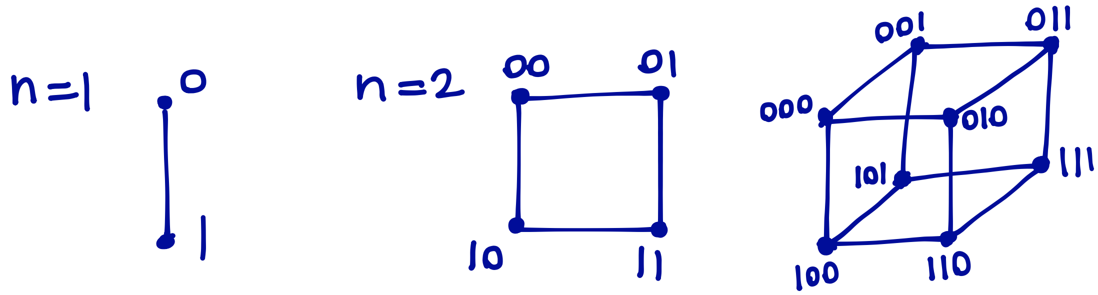
**3.**[10 points] A soccer team consists of 10 players (beside the goal keeper). There are three positions of defense, midfield and offense. Suppose there are $x_1 > 0$ players in defense, $x_2 > 0$ players in midfield and $x_3 > 0$ in offense. How many ways one can choose a formation for the team (i.e. a choice of $x_1, x_2, x_3 > 0$ with $x_1 + x_2 + x_3 = 10$)?

$$\binom{9}{2} = \frac{9!}{2!\,7!}$$

(Discussed several times in class).

**4.** For any $n \in \mathbb{N}$ consider the *hypercube graph* $H_n$ as follows: the vertices of $H_n$ are binary sequences of length $n$ (i.e. a sequence $(a_1, \ldots, a_n)$ where $a_i = 0, 1$). Two binary sequences $v = (a_1, \ldots, a_n)$, $w = (b_1, \ldots, b_n)$ are adjacent if and only if they differ exactly at one position (i.e. if there exists $1 \leq i \leq n$ such that $a_i \neq b_i$ and $a_j = b_j$ for all $j \neq i$).

**(a).**[3 points] Draw the graphs $H_1$, $H_2$ and $H_3$.



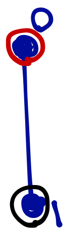

**(b).**[7 points] How many edges and vertices does $H_n$ have?

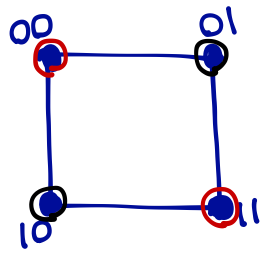
# of vertices = # of binary seq. of length n $= 2^n$.

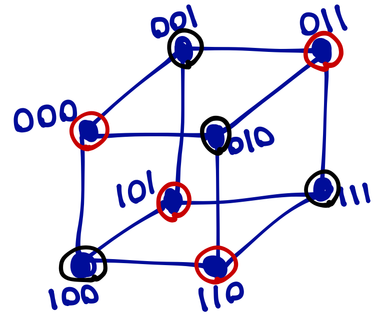
Note that each vertex $v=(a_1 \cdots a_n)$ can be changed in any of the n positions $a_1, \ldots, a_n$. So degree of each vertex is n.

Thus $2|E| = n2^n \Rightarrow |E| = n2^{n-1}$.

**5.(a)**[5 points] Let $H_n$ be the hypercube graph as in the previous problem. Show that $H_1$, $H_2$ and $H_3$ can be colored with 2 colors (vertex coloring).




**(b).**[5 points] Prove that for any $n \geq 1$ we have $\chi(H_n) = 2$, i.e. $H_n$ can be colored with 2 colors (you can prove it directly or you can use induction).

Color each vertex $v = (a_1 \cdots a_n)$ based on its parity

i.e. how many of its bits are 1.

- If $v$ has even number of 1's then color it $\boxed{\text{black}}$.

- If $v$ . . odd . . . . . . . . . . . . . . . . $\boxed{\text{red}}$.

By the way the graph $H_n$ is defined no two adjacent vertices have the same parity & hence the above is a valid vertex coloring.

**6.**[2 points] (Bonus) Draw (cartoon of) Euler crossing one of Königsberg bridges!