Solutions

Department of Mathematics

University of Pittsburgh

MATH 2370

Midterm 1, Fall 2015

Instructor: Kiumars Kaveh

Last Name: **Student Number:**

First Name:

TIME ALLOWED: 50 MINUTES. TOTAL: 50
NO AIDS ALLOWED. WRITE SOLUTIONS ON THE SPACE PROVIDED.

Question	Mark
1	/10
2	/10
3	/10
4	/10
5	/10
TOTAL	/50

1(a).[5 points] Give the definitions of the following: dual X' of vector space X, and dual T' of a linear map $T : X \to U$ (where X, U are vector spaces).

$X' = \{ \ell : X \to K \mid \ell \text{ linear} \}$

X' vec. space over K with addition & scalar multi. of functions.

$T' : U' \to X'$ linear map defined by: $T'(\ell) = \ell \circ T$.

(b).[5 points] Give the definition of the following: the direct sum of two vector spaces X and U.

$X \oplus U = \{ (x,u) \mid x \in X, u \in U \}$

$X \oplus U$ vec. space with Componentwise addition & scalar multi.

2.[10 points] Let $T : X \to U$ be a linear map where X, U are vector spaces and X is finite dimensional. Prove that $\dim(N_T) + \dim(R_T) = \dim(X)$.

<u>1st proof</u> Let $\{b_1, \dots, b_m\}$ be a basis for N_T, where $m = \dim(N_T)$.

Extend this to a basis $\{b_1, \dots, b_m, x_1, \dots, x_{n-m}\}$ for X, where $n = \dim(X)$.

We claim that $\{T(x_1), \dots, T(x_{n-m})\}$ is a basis for R_T.

Firstly, $\{T(x_1), \dots, T(x_{n-m})\}$ is lin. ind. because if $\sum_{i=1}^{n-m} c_i T(x_i) = 0 \Rightarrow T(\sum_{i=1}^{n-m} c_i x_i) = 0$

$\Rightarrow \sum_{i=1}^{n-m} c_i x_i \in N_T \Rightarrow \sum_{i=1}^{n-m} c_i x_i - \sum_{j=1}^{m} d_j b_j$ for some scalars d_j. Since $\{b_1, \dots, b_m, x_1, \dots, x_{n-m}\}$ is lin. ind. it follows that $c_1 = \dots = c_{n-m} = 0$. i.e. $\{T(x_1), \dots, T(x_{n-m})\}$ is lin. ind.

Secondly, $\{T(x_1), \dots, T(x_{n-m})\}$ spans R_T, because if $y = T(x) \in R_T$ then $x = \sum_{j=1}^{m} d_j b_j + \sum_{i=1}^{n-m} c_i x_i$. $\Rightarrow y = T(x) = \cancel{\sum_j d_j T(b_j)} + \sum_i c_i T(x_i) = \sum_i c_i T(x_i)$.

It follows that $\dim(R_T) = n-m$ as required.

<u>2nd proof</u> We show that $X/N_T \cong R_T$. The iso<u>ism</u> is given by $\tilde{T} : x + N_T \longmapsto T(x)$. To show $\tilde{T}$ is well-defined note that if $x_1 + N_T = x_2 + N_T$ then $x_1 - x_2 = n \in N_T$. $\Rightarrow T(x_1) - T(x_2) = T(n) = 0 \Rightarrow \tilde{T}(x_1 + N_T) = \tilde{T}(x_2 + N_T)$.

It is clear that $\tilde{T}$ is onto. To show that $\tilde{T}$ is one-to-one note that $N_{\tilde{T}} = \{x + N_T \mid x \in N_T\} = \{0 + N_T\} = \{0\}$. So $\tilde{T}$ is one-to-one.

Finally we know (from a previous thm.) that $\dim(X/N_T) = \dim(X) - \dim(N_T)$.

This finishes the proof.

3.[10 points] Let X be a finite dimensional vector space. Prove that a linear map $T : X \to X$ is one-to-one if and only if it is onto.

(because $R_T \overset{\text{subspace}}{\subset} X$)

First note that T is onto $\iff$ $R_T = X \iff \dim R_T = \dim X$

Also T is one-to-one $\iff$ $N_T = \{0\} \iff \dim N_T = 0$.

It is clear that if T is one-to-one then $N_T = \{0\}$.

Conversely if $N_T = \{0\}$ & $T(x_1) = T(x_2) \Rightarrow T(x_1 - x_2) = 0 \Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2$.

Thus T one-to-one.

Now since $\dim N_T + \dim R_T = \dim X$ we have:

$\dim N_T = 0 \iff \dim R_T = \dim X$. This finishes the proof.

(Remember: $\begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ = Col. vec. $x_1C_1 + \cdots + x_nC_n$.)

4.[10 points] Let p be a permutation of $\{1, \ldots, n\}$ and let P be the corresponding permutation matrix i.e. the i-th column of the matrix P is the standard basis element $e_{p(i)}$. In other words, P is obtained from the identity matrix by permuting the columns using p. Let D be a diagonal matrix:

$$D = diag(d_1, \ldots, d_n).$$

Show that the matrix $P^{-1}DP$ is again diagonal and in fact it is equal to $diag(d_{p(1)}, \ldots, d_{p(n)})$.

$\forall i = 1 \cdots n$, the i-th Col. of P is $e_{p(i)}$. It follows that $\forall j = 1 \cdots n$, the j-th row of P is $e_{p^{-1}(j)}$. Applying this to the permutation p^{-1} we see $\forall j = 1 \cdots n$, the j-row of P^{-1} is $e_{p(j)}$.

Direct calculation shows that:

$$\begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}\begin{bmatrix} e_{p(1)} & \cdots & e_{p(n)} \end{bmatrix} = \begin{bmatrix} d_{p(1)}e_{p(1)} & \cdots & d_{p(n)}e_{p(n)} \end{bmatrix}$$

And hence:

$$P^{-1}DP = \begin{bmatrix} e_{p(1)} \\ \vdots \\ e_{p(n)} \end{bmatrix}\begin{bmatrix} d_{p(1)}e_{p(1)} & \cdots & d_{p(n)}e_{p(n)} \end{bmatrix} = \begin{bmatrix} d_{p(1)} & & 0 \\ & \ddots & \\ 0 & & d_{p(n)} \end{bmatrix}$$

5.[10 points] Let P_1, P_2 be two projections from a finite dimensional linear space X into itself. Suppose $R_{P_1} = N_{P_2}$, is it always true that $P_1P_2 = 0$? Prove it or provide a counter example.

$R_{P_1} = N_{P_2}$ implies that $P_2P_1 = 0$.

It is not in general true that $P_1P_2 = 0$.

<u>Example</u> Let $X = \mathbb{R}^2$.

Let $P_1 =$ Proj. on x-axis parallel to y-axis ($R_{P_1} = \langle e_1 \rangle$, $N_{P_1} = \langle e_2 \rangle$)

Matrix of $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. (Easily checked $P_1^2 = P_1$).

Let $P_2 =$ Proj. on the line $x=y$ parallel to x-axis. ($N_{P_2} = \langle e_1 \rangle$, $R_{P_2} = \langle e_1 + e_2 \rangle$).

Clearly, P_2 proj. i.e. $P_2^2 = P_2$. Also $R_{P_1} = N_{P_2} =$ x-axis $= \langle e_1 \rangle$.

But $R_{P_2} \not\subset N_{P_1}$ because $R_{P_2} = \langle e_1 + e_2 \rangle$ & $N_{P_1} = \langle e_2 \rangle$. So $P_1P_2 \neq 0$.

Matrix of $P_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.