Solutions

Department of Mathematics

University of Pittsburgh

MATH 2371

Midterm, Spring 2016

Instructor: Kiumars Kaveh

Last Name: **Student Number:**

First Name:

TIME ALLOWED: 50 MINUTES. TOTAL: 50
NO AIDS ALLOWED. WRITE SOLUTIONS ON THE SPACE PROVIDED.

Question	Mark
1	/10
2	/10
3	/10
4	/10
5	/10
6	/2
TOTAL	/50 + 2

1.[10 points] Find the exponential of the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$A = \overbrace{\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}}^{D} + \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}^{N}$$

$D = I$ $\quad IN = NI = N.$

N nilpotent $\quad N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$N^3 = 0.$

$$e^A = e^{I+N} = e^I e^N = e\left(I + N + \frac{N^2}{2}\right) = e\begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e & e & e/2 \\ 0 & e & e \\ 0 & 0 & e \end{pmatrix}.$$

2.[10 points] Let A be a real symmetric matrix (hence self-adjoint).

(a) Show that A^2 is a ~~positive~~ non-negative matrix (in particular it is a symmetric matrix).

(b) Show that if A is a positive matrix then there exists Q such that $QAQ^* = I$ (Q is not necessarily orthagonal).

a) Firstly $(A^2)^T = (AA)^T = A^T A^T = AA = A^2$, so A^2 symm. $D = \text{diag}(\lambda_1, \dots, \lambda_n)$

A symm. $\Rightarrow \exists Q$ orthogonal s.t. $Q^T A Q = D$ diagonal with real entries.

Then $D^2 = Q^T A^2 Q$, so the diagonal entries of D^2 i.e. $(\lambda_1^2, \dots, \lambda_n^2)$ are eigenvalues of A^2. But $\lambda_i^2 \geq 0 \Rightarrow A^2$ non-negative matrix.

b) $A > 0 \Rightarrow A$ has a square root $A^{1/2}$ which is also positive & hence invertible.

Put $Q = (A^{\frac{1}{2}})^{-1}$. Then $A^{-\frac{1}{2}} A A^{-\frac{1}{2}} = I$.

3.[10 points] Let $A(t)$ be a matrix valued function such that $A(0) = I$ and $A(t)$ is a unitary matrix for any t. Show that the derivative $A'(0)$ is an anti-self-adjoint matrix.

$$A(t)\,A^*(t) = I \Rightarrow \frac{d}{dt}\Big|_{t=0}\left(A(t)\,A^*(t)\right) = A'(0)\,A^*(0) + A(0)\,A^{*\prime}(0) = 0$$

$$A'(0)\cdot I + I\cdot\left(A'(0)\right)^* = 0 \Rightarrow A'(0) + \left(A'(0)\right)^* = 0$$

$$\Rightarrow A'(0) = -A'(0) \Rightarrow A'(0) \text{ is anti-self-adj.}$$

4.[10 points] Suppose A is a real positive matrix. Suppose v is a vector such that $||Av|| = ||A||||v||$. Show that v is an eigenvector of A corresponding to the largest eigenvalue of A.

Since A self-adj. $\Longrightarrow$ $||A|| = r(A) =$ largest eigenvalue

Let $\{v_1, \dots, v_n\}$ be an orthonormal basis of eigenvec. of A.

Let λ_i be the eigenval. corr. to v_i. Also assume $\lambda_1 \leq \cdots \leq \lambda_n$.

Let us write $v = \sum_{i=1}^{n} c_i v_i$. Then:

because $\{v_1 \cdots v_n\}$ orth. basis

$$||Av||^2 = ||\sum_i c_i A v_i||^2 = ||\sum_i c_i \lambda_i v_i||^2 = \sum_i |c_i \lambda_i|^2.$$

$$||v||^2 = ||\sum_i c_i v_i||^2 = \sum_i |c_i|^2.$$

$||Av|| = ||A|| \, ||v|| \Longrightarrow \sum_i |c_i \lambda_i|^2 = \sum_i |c_i \lambda_n|^2 \Longrightarrow c_i = 0$ whenever $\lambda_i < \lambda_n$.

This shows that v is an eigenvec. with eigenvalue λ_n.

5.[10 points] Let A, B be $n \times n$ self-adjoint matrices. Suppose that $A < B$ and $AB = BA$. Show that $e^A < e^B$.

$AB = BA \Rightarrow A, B$ can be diagonalized using the same unitary

matrix U: $\quad U^*AU = D_1 = \mathrm{diag}(\lambda_1, \dots, \lambda_n).$
$U^*BU = D_2 = \mathrm{diag}(\mu_1, \dots, \mu_n).$ $\Bigg\}$ $U^*(B-A)U = D_2 - D_1 \Longrightarrow$
eigenval. of $B-A = \{\mu_1 - \lambda_1, \dots, \mu_n - \lambda_n\}$

$B - A > 0 \Longrightarrow \underset{\forall i}{\mu_i - \lambda_i > 0} \Longrightarrow \underset{\forall i}{\mu_i > \lambda_i} \Longrightarrow e^{\mu_i} > e^{\lambda_i}.$

$\Rightarrow U^*(e^B - e^A)U = e^{D_2} - e^{D_1} = \mathrm{diag}(e^{\mu_1} - e^{\lambda_1}, \dots, e^{\mu_n} - e^{\lambda_n}).$

6.[2 points] (Bonus) Draw (cartoon of) a prelim exam torturing a student!