Solutions

Department of Mathematics

University of Pittsburgh
MATH 1050 (Combinatorics)
Midterm 2 (Fall 2015)

Last Name: **Student Number:**

First Name:

TIME ALLOWED: 50 MINUTES. TOTAL: 50
NO AIDS ALLOWED. WRITE SOLUTIONS ON THE SPACE PROVIDED.

Question	Mark
1	/12
2	/12
3	/10
4	/8
5	/10
6	/2
TOTAL	/50

1(a).[6 points] Give the definition of a planar graph and state Euler's formula about the number of vertices, edges and faces of a planar graph.

A graph is planar if it can drawn on the plane where vertices are represented by points & edges are (curved) lines connecting the vertices. Moreover two edges can only intersect at a common vertex.

Euler's formula: G planar graph $n - e + f = 2$ where $n = |V|$, $e = |E|$, f = # of regions obtained in the plane after drawing G.

(b).[6 points] State the Ramsey theorem and define the Ramsey number $R(m, n)$.

Given integers $n, m > 0$

<u>Thm</u> $\exists$ integer $r > 0$ such that if we color the edges of the complete graph K_r with two colors black & white (in an arbitrary way) then either in K_r we can find a black complete graph with n vertices or in K_r we can find a white complete graph with m vertices.

<u>Def.</u> Given $n, m > 0$ the Ramsey number $R(m,n)$ is the smallest $r > 0$ in the above theorem.

2(a).[6 points] Consider the sequence a_n defined recursively by $a_0 = 1$, $a_1 = 0$, $a_n + 3a_{n+1} = a_{n+2}$. Find a formula for the generating function $F(x) = \sum_{n=0}^{\infty} a_n x^n$. That is, express this function as a rational function (quotient of two polynomials in x).

$$a_n + 3a_{n+1} = a_{n+2} \Rightarrow a_n x^{n+2} + 3a_{n+1} x^{n+2} = a_{n+2} x^{n+2} \Rightarrow$$

$$\sum_{n=0}^{\infty} a_n x^{n+2} + 3\sum_{n=0}^{\infty} a_{n+1} x^{n+2} = \sum_{n=0}^{\infty} a_{n+2} x^{n+2} \Rightarrow$$

$x^2 F(x) + 3x(F(x) - 1) = F(x) - 1$. Solve for $F(x)$:

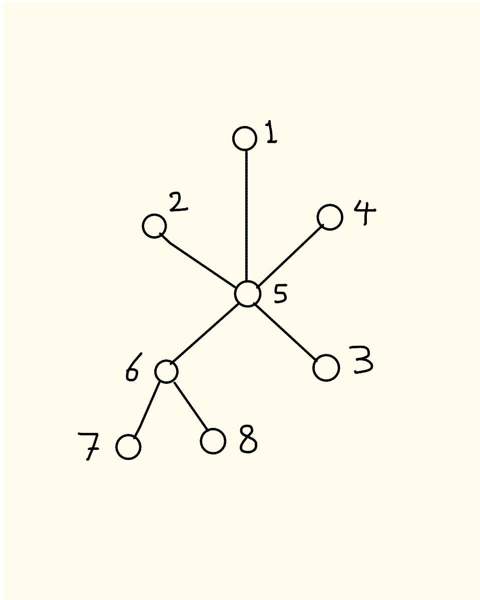
$$(x^2 + 3x - 1) F(x) = 3x - 1 \Rightarrow F(x) = \frac{3x-1}{x^2+3x-1}.$$

(b)[6 points] Let a_n denote the number of ways n can be written as $n = x_1 + x_2 + x_3 + x_4$ where the x_i are integers and $0 \leq x_1$, $1 \leq x_2$, $2 \leq x_3$ and $0 \leq x_4 \leq 1$. Find an expression for the generating function $F(x)$ of the sequence a_n.

$$F(x) = \overbrace{(1 + x + x^2 + \cdots)}^{0 \leq x_1} \overbrace{(x + x^2 + \cdots)}^{1 \leq x_2} \overbrace{(x^2 + x^3 + \cdots)}^{2 \leq x_3} \overbrace{(1 + x)}^{0 \leq x_4 \leq 1}$$

$$F(x) = \frac{1}{1-x} \cdot \frac{x}{1-x} \cdot \frac{x^2}{1-x} \cdot (1+x) = \frac{(1+x)x^3}{(1-x)^3}.$$

3(a).[5 points] Write the Prüfer code for the following tree.



5 5 5 5 6 6 $\rightarrow$ Seq. of length $n-2=6$.

(b).[5 points] Let G be a (simple) graph. Use Pigeon Hole Principle to show that there are two vertices in G which have the same degree. Hint: let $|V|=n$ then the degree of each vertex is at most $n-1$.

Without loss of generality we can assume G does not have any degree $\underline{0}$ vertices (if it has we just ignore them).

Then $\forall v \in V \quad 1 \leqslant \deg(v) \leqslant n-1$, because any vertex is conn. to at most $n-1$ other vertices. Now by Pigeon Hole Principle two vertices should have same degree (because there are n vertices)

4.[8 points] Suppose we have n balls numbered $1, \ldots, n$. How many ways we can color them with three colors blue, white and red such that all the three colors are used. Equivalently how many surjective functions are there from the set $\{1, \ldots, n\}$ to the set $\{blue, white, red\}$. (Hint: as we discussed in class the answer involves using the principle of Inclusion-Exclusion, you don't need to simplify your final answer).

\# of all functions from $\{1, \cdots, n\}$ to $\{b, w, r\} = 3^n$

. . . . $\{b, w\} = 2^n$

. . . . $\{w, r\} = 2^n$

. . . . $\{b, r\} = 2^n$

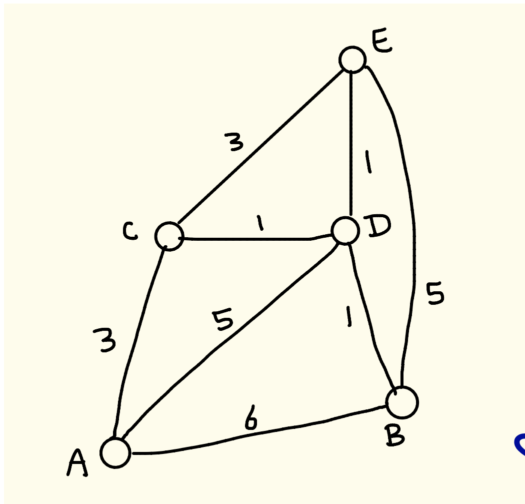
. . . . $\{r\} = 1$

. . . . $\{b\} = 1$

. . . . $\{w\} = 1$

\# of surj. functions from $\{1 \cdots n\}$ to $\{b, r, w\}$ $= 3^n - 2^n - 2^n - 2^n + 1 + 1 + 1 = 3^n - 3 \cdot 2^n + 3$.

5.[10 points] Consider the following weighted graph. Run the Dijkstra algorithm to find the shortest path from vertex A to vertex E. Write the values of the lists δ and σ at each step.



σ	δ: A	B	C	D	E
A	0	∞	∞	∞	∞
A, C	×	6	3	5	∞
A, C, D	×	6	×	4	6
A, C, D, E done	×	6	×	×	5
A, C, D, E, B	×	6	×	×	×

6.[2 points] (Bonus) Draw (cartoon of) a mathematician crying because he/she found a mistake in his/her proof!