Solutions

Department of Mathematics

University of Pittsburgh

MATH 2370

Midterm 2, Fall 2015

Instructor: Kiumars Kaveh

Last Name: **Student Number:**

First Name:

TIME ALLOWED: 50 MINUTES. TOTAL: 50
NO AIDS ALLOWED. WRITE SOLUTIONS ON THE SPACE PROVIDED.

Question	Mark
1	/10
2	/10
3	/12
4	/10
5	/8
6	/4
TOTAL	/50

1(a).[5 points] Give the definitions of a symmetric bilinear form and a scalar product on a vector space V.

Let V vec. space over a field K.

$B: V \times V \longrightarrow K$ is a bilinear form if B is linear in both arguments.

$B: V \times V \longrightarrow K$ is symm. if $\forall v, w \in V$ $B(v,w) = B(w,v)$.

Let V be a vec. space over $\mathbb{R}$.

A symm. bilinear form $B: V \times V \longrightarrow \mathbb{R}$ is a scalar product if $\forall v \in V$ $B(v,v) \geq 0$,

& moreover, $B(v,v) = 0 \iff v = 0$.

(b).[5 points] Give the definition of a generalized eigenvector. State the Spectral Theorem.

$T: V \longrightarrow V$ lin. map.

$0 \neq v \in V$ is a generalized eigenvec. with eigenvalue λ if for some integer $m > 0$

we have $(T - \lambda I)^m (v) = 0$.

Spectral theorem

V is the direct sum of generalized eigenspaces for T.

2. Let V be a finite dimensional vector space and let $P : V \to V$ be a linear projection i.e. $P^2 = P$.

(a)[5 points] Prove that $V = N_P \oplus R_P$ where N_P and R_P are the null space and range of P respectively.

Let $w = P(v) \in R_P$. Suppose $w \in N_P \Rightarrow P(P(v)) = 0$, but $P^2(v) = P(v) \Rightarrow w = 0$. That is

$R_P \cap N_P = \{0\}$. On the other hand $\dim N_P + \dim R_P = \dim V$. This implies that $V = N_P \oplus R_P$.

(b)[5 points] Show that P is diagonalizable.

Let $n = \dim V$, $k = \dim N_P$ & $n-k = \dim R_P$.

Let $\{b_1, \dots, b_k\}$ basis for N_P & $\{b_{k+1}, \dots, b_n\}$ basis for R_P.

Clearly $\forall v \in N_P$, $v \neq 0$: $P(v) = 0 \Rightarrow v$ eigenvec. with eigenvalue $\underline{0}$.

$\forall v \in R_P$, $v \neq 0$: $P(v) = v \Rightarrow v$ eigenvec. with eigenvalue $\underline{1}$.

The basis $\{b_1, \dots, b_n\}$ for V Consists of eigenvec. $\Rightarrow P$ diagonalizable.

3(a).[8 points] Find the characteristic polynomial of A. Find the minimal polynomial of A. Find the generalized eigenspaces of A.

A block diagonal

$$A = \begin{bmatrix} \boxed{\begin{matrix}0 & 1\\ 1 & 0\end{matrix}} & \begin{matrix}0 & 0\\ 0 & 0\end{matrix} & \begin{matrix}0 & 0\\ 0 & 0\end{matrix} \\ \begin{matrix}0 & 0\\ 0 & 0\end{matrix} & \boxed{\begin{matrix}1 & 0\\ 1 & 1\end{matrix}} & \begin{matrix}0 & 0\\ 0 & 0\end{matrix} \\ \begin{matrix}0 & 0\\ 0 & 0\end{matrix} & \begin{matrix}0 & 0\\ 0 & 0\end{matrix} & \begin{matrix}\boxed{0} & 0\\ 0 & \boxed{1}\end{matrix} \end{bmatrix}.$$

char. poly. of $\begin{bmatrix}0&1\\1&0\end{bmatrix} = t^2-1 = (t-1)(t+1)$

min. poly .... $= (t-1)(t+1)$

eigenvec. $\to v_1 = \begin{bmatrix}1\\1\end{bmatrix}$ $\lambda_1=1 \rightsquigarrow e_1+e_2$

$v_2 \begin{bmatrix}1\\-1\end{bmatrix}$ $\lambda=-1 \rightsquigarrow e_1-e_2$

char. poly.:

$P_A(t) = (t-1)(t+1)(t-1)^2 t\,(t-1) = (t-1)^4(t+1)\,t.$

char. poly. of $\begin{bmatrix}1&0\\1&1\end{bmatrix} = (t-1)^2.$

min. poly of .... $= (t-1)^2.$

Only eigen vec. $= \begin{bmatrix}0\\1\end{bmatrix}$

gen. eigen space of 1 $\longrightarrow$ span$\{e_3, e_4\}$

min. poly.:

$m_A(t) = (t-1)^2(t+1)\,t.$

eigenvalues: $1, -1, 0$

Gen. eigenspace of $\underline{\underline{0}}$ = span$\{e_5\}$

Gen. eigenspace of 1 = span$\{e_1+e_2, e_3, e_4, e_6\}$

Gen. eigenspace of -1 = span$\{e_1-e_2\}$

(b)[4 points] Let A be a square matrix. Suppose the characteristic polynomial of A is $(t-1)^3(t+1)$ and the minimal polynomial of A is $(t-1)(t+1)$. Find the Jordan canonical form of A. First deg. char. poly. $= 4 \Rightarrow A$ is 4×4.

min. poly. $= (t-1)(t+1) \Longrightarrow$ eigenvalues are 1 & -1 & each one has index $\underline{\underline{1}}$.

So the sizes of Jordan blocks are $\underline{\underline{1}}$. Thus the Jordan form is:

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$$

4(a).[5 points] Let A be a 3×2 matrix and B be a 2×3 matrix. Show that $\det(AB) = 0$.

Let C_1 & C_2 be the Col. of A i.e. $A = \begin{bmatrix} | & | \\ C_1 & C_2 \\ | & | \end{bmatrix}$.

Then the Col. of AB are lin. Comb. of C_1 & C_2. Since AB has $\underline{\underline{3}}$ Col. & they are lin. Comb. of $\underline{2}$ vec. C_1 & C_2 $\Rightarrow$ they should be lin. dep. $\Longrightarrow$ $\det(AB) = 0$.

(b)[5 points] Let $\{e_1, \ldots, e_4\}$ be the standard basis for $\mathbb{R}^4$. Let P be the 4×4 permutation matrix such that $Pe_i = e_{5-i}$ for every $i = 1, \ldots, e_4$ (i.e. $Pe_1 = e_4$, $Pe_2 = e_3$ etc.). Find the determinant of P.

Matrix of P looks like:

$$\begin{bmatrix} 0 & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & 0 \end{bmatrix}$$

Switching 1st & 4th col. & then switching 2nd & 3rd col. we get the identity matrix I & $\det(I) = 1$.

Any switching of columns multiplies the det by (-1). So $\det(P) = (-1)(-1) = 1$.

5.[8 points] Suppose A and B are complex 3×3 nilpotent matrices with $A^2 = B^2 = 0$. Prove that A and B have a common eigenvector.

Since A & B are nilpotent their only eigenvalue is $\underline{\underline{0}}$.

We claim that $\dim N_A$ & $\dim N_B$ are ≥ 2.

Suppose $\dim N_A = 1$. Then the Jordan form of A has only 1 block & should be $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

But then $A^2 \neq 0$ (rather $A^3 = 0$) which is a contradiction. Similarly $\dim N_B = 1$ is not possible.

Now if $N_A \cap N_B = \{0\} \Rightarrow \dim(N_A + N_B) \geq 2 + 2 = 4$ which is not possible because the whole space is 3 dim.

Thus $N_A \cap N_B \neq \{0\} \Rightarrow A$ & B have a nonzero eigenvec.

6.[4 bonus points] Let V and W be finite dimensional vector spaces over $\mathbb{R}$. Let $B : V \times W \to \mathbb{R}$ be a non-degenerate bilinear form. That is, B is linear in both arguments and the following holds: if $B(v, w) = 0$ for all $w \in W$ then $v = 0$, and similarly if $B(v, w) = 0$ for all $v \in V$ then $w = 0$.
Prove that $\dim(V) = \dim(W)$.

Consider the linear map $v \longmapsto l_v = B(v, *)$. $\quad V \longrightarrow W'$

That is, l_v is the lin. function on W defined by $l_v(w) = B(v, w)$.

Since B is non-degenerate the map $v \longmapsto l_v$ has null space $= \{0\}$,

& thus is one-to-one. So we have a one-to-one lin. map from V to W'.

This shows that $\dim V \leq \dim W' = \dim W$.

Switching the roles of V & W we see $\dim W \leq \dim V$.

This finishes the proof.