List of topics to review for linear algebra prelim exam

Kiumars Kaveh

April 25, 2016

- Vector space, subspace, independent set, spanning set, basis, isomorphism of vector spaces, a basis for an n-dimensional vector space V gives an isomorphism between V and K^n (where K is the base field)
- Direct sum of two vector spaces, direct sum of two subspaces of a vector space
- Linear map, matrix of a linear map with respect to a basis, change of basis matrix, the matrices M_1, M_2 of a linear map T with respect to two different bases B_1, B_2 are similar matrices
- If $V = W \oplus U$ then we have a projection $P : V \to W$ (projection parallel to U), conversely if P is such that $P^2 = P$ (i.e. a projection) then $V = R_P \oplus N_P$
- Dual V' of a vector space V, dual T' of a linear map T. Dual of a linear map corresponds to the transpose of the corresponding matrix
- There is a natural pairing $V \times V' \to K$ between V and it dual space V'. It is a bilinear map If B is a basis for V there is a natural dual basis B' for V' called the dual basis such that the pairing between the elements of B and B' is the identity matrix.
- Nilpotent matrix, permutation matrix
- Determinant, geometric interpretation of determinant as (signed) volume, definition of determinant as an alternating n-linear map, formula for determinant in terms of sum over all permutations, determinant of special matrices: diagonal, upper/lower triangular, block diagonal, Cramer's rule for solving a system of equations using determinants, formula for inverse of a matrix in terms of determinants, Laplace row/column expansion of a determinant, $\det(AB) = \det(A)\det(B)$, $\det(A) = \det(A^T)$, trace, $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, trace is invariant under conjugation
- Eigenvalues, eigenvectors, generalized eigenvectors, characteristic polynomial, minimal polynomial, first and last coefficients of characteristic polynomial in terms of trace and determinant

- Spectral Theorem (p. 70 in Lax), how to find (generalized) eigenvectors and eigenvalues of a give matrix, index of a generalized eigenvector, Jordan decomposition
- How to find Jordan form if the characteristic polynomial and the minimal polynomial are given
- A and A^T are similar matrices
- Bilinear form $B : V \times V \to K$ (where K is the base field), matrix M of a bilinear form B with respect to a choice of a basis for V, if we change the basis then the matrix M of B changes to $S^T MS$ where S is the change of basis matrix
- Euclidean structures, scalar product, orthonormal basis, Gram-Schmidt orthonormalization, orthogonal matrix
- Adjoint linear map, orthogonal projection, reflection with respect to a hyperplane (formula)
- Theorem of least squares, i.e. given matrix A and vector p find x such that $||Ax - p||^2$ is minimum
- Hermitian structures, Hermitian product, Hermitian matrix, unitary matrix
- Operator norm, Hilbert-Schmidt norm, spectral radius, Gelfand's formula (p. 334 in Lax), Theorem that if A is invertible and $||A - B|| < 1/||A^{-1}||$ then B is invertible
- Self-adjoint operator, normal matrix, every normal matrix is orthogonally (unitarily) diagonalizable
- Quadratic form, matrix of a quadratic form, diagonalizing a quadratic form by a linear change of variables, how to distinguish that a curve defined in the plane by a quadratic equation is an ellipse or hyperbola, Sylvester's law of inertia about number of positive, negative and zero eigenvalues of a symmetric real matrix (or its associated quadratic form)
- Rayleigh quotient and Courant-Hilbert Minimax Principle about the i-th eigenvector of a Hermitian matrix (p. 116 in Lax)
- Calculus of matrix valued functions, rules of differentiation for matrix product and matrix inverse, derivative of trace and determinant
- Matrix exponential, $e^{A+B} = e^A e^B$ if $AB = BA$, exponential of a diagonal matrix
- Every matrix can be written as $A = D + N$ with $DN = ND$ where D is diagonalizable and N is nilpotent (corollary of Jordan canonical form), using this to find exponential of a matrix

- Positive matrices (another commonly used term is positive-definite matrix), positive matrices form a convex cone in the (real) vector space of all self-adjoint matrices, Sylvester's criterion for a matrix to be positive (in terms of determinants of principal minors)
- Every positive matrix has a unique square root which is also positive, Gram matrix, a matrix is nonnegative if and only if it is a Gram matrix
- Matrix inequalities e.g. if $0 < A < B$ then $A^{-1} > B^{-1}$, determinant is a log-concave function (p. 154 in Lax), estimating determinant of a positive matrix by product of its diagonal entries (p. 155 in Lax)
- Polar Decomposition and Singular Values Decomposition (p. 169-170 in Lax), QR factorization (p. 262 in Lax, it is basically Gram-Schmidt orthonormalization)

Tpoics discussed in the course but will not be in the prelim exam:

- Gershgorin circle theorem (discussed in the recitation)
- Brief discussion of multi-linear algebra: tensor product of vector spaces, symmetric power and exterior power
- Entry-wise positive matrices, Perron's theorem (p. 237 in Lax)
- Brief discussion of convexity and the König-Birkhoff-Von Neumann theorem (p. 198 in Lax)