MATH2810 ALGEBRAIC GEOMETRY NOTES

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Contents

The ground field $\mathbf{k}$ is assumed to be an algebraically closed field.

1. Affine Algebraic Geometry

Definition 1.1. Let $f_1, \ldots, f_r \in \mathbf{k}[x_1, \ldots, x_n]$. Define

$$V(f_1, \ldots, f_r) = \{x \in \mathbf{k}^n \mid f_1(x) = \cdots = f_r(x) = 0\}.$$

It can be shown that $V(f_1, \ldots, f_r) = V(I)$ where I is the ideal generated by the polynomials f_i and

$$V(I) = \{x \in \mathbf{k}^n \mid f(x) = 0 \ \forall f \in I\}.$$

Any set of the form $V(f_1, \ldots, f_n)$ is called an *algebraic set* or an *affine algebraic variety*. Conversely, by Hilbert's basis theorem any ideal of $\mathbf{k}[x_1, \ldots, x_n]$ is finitely generated and hence for any ideal I we can find $f_1, \ldots, f_r \in I$ such that $V(I) = V(f_1, \ldots, f_r)$. Also for a subset $X \subset \mathbf{k}^n$ we put

$$I(X) = \{f \in \mathbf{k}[x_1, \ldots, x_n] \mid f(x) = 0, \ \forall x \in X\}.$$

Theorem 1.2 (Hilbert's Nullstellensatz (weak formulation))**.** *If $I \subset \mathbf{k}[x_1, \ldots, x_n]$ and $I \neq \mathbf{k}[x_1, \ldots, x_n]$, then $V(I) \neq \emptyset$.*

Proposition 1.3 (Zariski Topology)**.** *One can define a topology on* $\mathbf{k}^n$ *where closed sets are algebraic, i.e. of the form* $V(I)$*. The Zariski topology is Noetherian (every increasing chain of open subsets stabilizes), compact, and not Hausdorff.*

An open subset of an affine variety is sometimes called a *quasi-affine variety.*

Definition 1.4. The radical of an ideal $I \subset \mathbf{k}[x_1, \ldots, x_n]$ is defined as follows:

$$\mathrm{rad}(I) = \{f \in \mathbf{k}[x_1, \ldots, x_n] \mid f^N \in I \text{ for some } N \in \mathbb{N}\}.$$

Then, $\mathrm{rad}(I)$ is an ideal and $\mathrm{rad}(\mathrm{rad}(I)) = \mathrm{rad}(I)$.

Theorem 1.5 (Hilbert's Nullstellensatz)**.** *Let* $I \subset \mathbf{k}[x_1, \ldots, x_n]$*. Then* $I(V(I)) = \mathrm{rad}(I)$.

Theorem 1.6 (Galois Correspondence)**.** *Let* P, Q *be partially ordered sets,* $f : P \to Q, g : Q \to P$ *be order-reversing maps. Suppose* $f \circ g, g \circ f \geq id$*. Then* f *and* g *give a 1-1 correspondence between* $g(Q)$ *and* $f(P)$*.*

In our case, $P =$ all ideals of $\mathbf{k}[x_1, \ldots, x_n]$, $Q =$ all subsets of $\mathbf{k}^n$, $f(I) = V(I)$ and $g(X) = I(X)$.

Definition 1.7 (Irreducible algebraic set)**.** An algebraic set is *irreducible* if it cannot be written as the union of two distinct non-empty closed subsets.

Proposition 1.8. $W = V(I)$ *is irreducible if and only if* I *is a prime ideal in* $\mathbf{k}[x_1, \ldots, x_n]$*.*

Proposition 1.9. *Every closed set* V *can be decomposed uniquely into a finite union of irreducible closed sets.*

Equivalently, every radical ideal can be uniquely represented as an intersection of prime ideals.

Definition 1.10. Let $V = V(I)$ where $I = \mathrm{rad}(I)$ is a radical ideal. Define the coordinate ring $\mathbf{k}[V]$ as follows:

$$\mathbf{k}[V] = \mathbf{k}[x_1, \ldots, x_n]/I.$$

If V is irreducible then I is prime and $\mathbf{k}[V]$ is an integral domain. In this case we denote the field of fractions of $\mathbf{k}[V]$ by $\mathbf{k}(V)$, and call it the *field of rational functions on* V.

Definition 1.11 (Dimension)**.** Let V be an irreducible variety. The dimension of V, $\dim(V)$ is equal to the transcendence degree of $\mathbf{k}(V)$ over $\mathbf{k}$. The dimension of a non-irreducible variety is the maximum of dimensions of its irreducible components.

Proposition 1.12 (Basic facts about dimension)**.** *A hypersurface* $V(f)$ *has dimension* $n - 1$*. If* V *is an irreducible algebraic set,* $Z \subset V, Z \neq V$*, then* $\dim(Z) < \dim(V)$*.*

Definition 1.13 (Krull dimension)**.** Let A be a commutative ring. Let

$$P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n \subsetneqq A,$$

be a sequence of prime ideals. We say that this sequence has *length* n. The *Krull dimension* of A is the supremum of lengths of sequences of prime ideals in A. The rings we deal with all have finite length, although in general it is possible for a Noetherian domain to have infinite Krull dimension.

We stated but did not prove the following theorem which relates the notion of Krull dimension with transcendence degree of the field of rational functions (see [Milne, Chapter 9], more specifically Corollary 5.9). The proof is based on the so-called Krull's *Hauptidealsatz*.

Theorem 1.14. *Let V be an irreducible affine algebraic variety of dimension d. Then the coordinate ring $\mathbf{k}[V]$ has Krull dimension d.*

Corollary 1.15. *Let V be an irreducible affine algebraic variety. Then dimension of V is equal to the length d of a maximal chain*

$$\emptyset \neq V_0 \subset \cdots \subset V_d = V$$

of closed irreducible subvarieties of V.

2. Sheaves

Definition 2.1 (Sheaf of algebras)**.** Let X be a topological space. Suppose the following conditions hold:

- For $U \subset X$ open, $\mathcal{F}(U)$ is a $\mathbf{k}$-algebra.
- If $U \subset V$ then we have a $\mathbf{k}$-algebra homomorphism $\alpha_V^U : \mathcal{F}(V) \to \mathcal{F}(U)$, such that for $W \subset U \subset V$,
$$\alpha_U^W \circ \alpha_V^U = \alpha_V^W.$$
- $\mathcal{F}(\emptyset) = 0$ and $\alpha_U^U = id$.
- For $U = \cup V_i$ open, if $\alpha_U^{V_i}(f) = 0$ for all i for some $f \in \mathcal{F}(U)$, then $f = 0$.
- For $U = \cup V_i$ open, $f_i \in \mathcal{F}(V_i)$, if $\alpha_{V_i}^{V_i \cap V_j}(f_i) = \alpha_{V_j}^{V_i \cap\, V_j}$ for all i, j, then $\exists f \in \mathcal{F}(U)$ such that $\alpha_U^{V_i}(f) = f_i$ for all i.

Then $(X, \mathcal{F})$ is called a *sheaf of algebras*.

Let $X \subset \mathbf{k}^n$ be an affine variety. Let $h \in \mathbf{k}[X]$. Define the basic open set

$$D(h) = X \setminus V(h).$$

The open sets $D(h)$ are basis for the Zariski topology.

Definition 2.2. Given $g, h \in \mathbf{k}[X]$,

$$p \mapsto \frac{g(p)}{h(p)}$$

defines a function $D(h) \to \mathbf{k}$. For a point $p \in X$, a function f defined on neighborhood of p, is called *regular at p* if in some neighborhood of p, it coincides with g/h for some $g, h \in \mathbf{k}[X]$ with $h(p) \neq 0$. A function f is said to be *regular on an open subset U* if it is regular at every $p \in U$. A function which is regular on some non-empty open subset of X is called a *rational function* on X.

The collection of regular functions is a $\mathbf{k}$-algebra. The collection of rational functions on X is a field extension of $\mathbf{k}$.

We denote the algebra of regular functions on an open set $U \subset X$ by $\mathcal{O}(U)$. This defines a sheaf of algebras on X which is denoted by $\mathcal{O}_X$ called the *sheaf of regular functions on* X or *structure sheaf of* X.

For each $p \in X$, the local ring of X at p is the germ of regular functions at p, that is the collection of (f, U) where $p \in U$ and f is regular on U. We consider (f, U) and (g, V) the same if $f = g$ on $U \cap V$. The following is the basic result about regular functions (see [Milne] or [Hartshorne]).

Theorem 2.3. *Let X be an irreducible affine variety.*

(a) $\mathcal{O}(X) \cong \mathbf{k}[X]$

(b) *For each $p \in X$ let $\mathfrak{m}_p \subset \mathbf{k}[X]$ be the ideal of functions vanishing at p. Then $p \mapsto \mathfrak{m}_p$ gives a one-to-one correspondence between the points of X and the maximal ideals of $\mathbf{k}[X]$.*

(c) *For each $p \in X$, $\mathcal{O}_{p,X} \cong \mathbf{k}[X]_{\mathfrak{m}_p}$, the localization of the ring $\mathbf{k}[X]$ at the maximal ideal $\mathfrak{m}_p$.*

(d) *The collection of rational functions on X is isomorphic to the quotient field $\mathbf{k}(X)$ of $\mathbf{k}[X]$.*

Definition 2.4 (Morphisms of sheaves)**.** $\phi : (X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ is a morphism if it is a continuous map from X to Y such that $\forall U \subset Y$ and $f \in \mathcal{F}_Y(U)$, $f \circ \phi \in \mathcal{F}_X(\phi^{-1}(U))$.

Definition 2.5 (Coordinate free definition of an affine algebraic variety)**.** A sheaf $(X, \mathcal{F})$ is an affine algebraic variety if it isomorphic to the sheaf of regular functions $\mathcal{O}(Y)$ on an algebraic set Y.

Definition 2.6 (Morphism of algebraic varieties)**.** $\phi : X \to Y$ is a morphism of algebraic varieties X and Y if it gives a morphism of the corresponding sheaves of regular functions $\mathcal{O}_X$ and $\mathcal{O}_Y$.

Theorem 2.7. *Let $X \subset \mathbf{k}^n$ and $Y \subset \mathbf{k}^m$ be affine algebraic varieties. Let $\phi : X \to Y$ be continuous in the Zariski topology. Then the following are equivalent:*

- *ϕ is a morphism*
- *$\phi = (\phi_1, \dots, \phi_m)$, $\forall i, \phi_i$ is a regualr function*
- *$\forall f \in \mathbf{k}[Y]$, $f \circ \phi \in \mathbf{k}[X]$*

ϕ is called a dominant map if $\phi(X)$ is dense in Y.

Theorem 2.8. *There is an equivalence of categories between the category of affine algebraic varieties and morphisms and "reduced" finitely generated $\mathbf{k}$-algebras and $\mathbf{k}$-algebra homomorphisms (reduced means without nilpotent element).*

3. Projective varieties

Definition 3.1 (Projective Variety)**.** Let $I \subset \mathbf{k}[x_0, \dots x_n]$ be a homogenous ideal. Take $V_{aff}(I) = \{(a_0, \dots, a_n) \mid f(a_0, \dots, a_n) = 0 \ \forall f \in I\} \subset \mathbb{A}^{n+1}$ and look at its image inside $\mathbb{P}^n$. We define $V(I)$ to be that image, and call it a *projective variety.*

Theorem 3.2. *V and I are inverse bijections between proper homogenous radical ideals in $\mathbf{k}[x_0, \ldots, x_n]$ and projective varieties in $\mathbb{P}^n$.*

Definition 3.3. Let $X \subset \mathbb{P}^n$ be a projective variety with homogeneous ideal $I \subset \mathbf{k}[x_0, \ldots, x_n]$. We call the quotient ring $\mathbf{k}[x_0, \ldots, x_n]/I$ the *homogeneous coordinate ring of the projective variety X*. Since the ideal I is homogeneous, the grading of polynomials by degree induces a grading on $\mathbf{k}[X]$.

Proposition 3.4. *The following hold for homogenous ideals I, J:*

- $I \subset J \Rightarrow V(J) \subset V(I)$
- $V(0) = \mathbb{P}^n$
- $V(I) = \emptyset \Leftrightarrow (x_0, \ldots, x_n) \subset \mathrm{rad}(I)$
- $V(IJ) = V(I \cap J) = V(I) \cup V(J)$
- $V(\sum I_i) = \cap_i V(I_i)$
- *I homogenous implies $\mathrm{rad}(I)$ is homogenous*

Thus, $V(I)$ form closed sets for a topology which we call the Zariski topology on $\mathbb{P}^n$. The topology induced on a projective variety $X \subset \mathbb{P}^n$ is called the Zariski topology on X.

Definition 3.5 (Quasi-projective variety)**.** An open subset of a projective variety is called a *quasi-projective variety.* Affine, quasi-affine and projective varieties are all quasi-projective.

Define $U_0 = \{(x_0 : \cdots : x_n) | x_o \neq 0\} \subset \mathbb{P}^n$ which is an open set, and define $H_\infty = \mathbb{P}^n \setminus U_0$. Then we have the following theorem:

Theorem 3.6. *U_0 is homeomorphic to $\mathbb{A}^n$ with respect to their Zariski topologies.*

From here, we can define regular functions, regular maps, local ring of a point and rational function analogously to the affine case. MreLet X, Y be quasi-projective varieties. A map $\phi : X \to Y$ is regular if ϕ is continuous, and for any open subset $U \subset Y$ and any $f \in \mathcal{O}(U)$, $f \circ \phi \in \mathcal{O}(\phi^{-1}(U))$.

The following are the basic facts about regular functions on projective varieties. For proofs see [Milne] and [Hartshorne].

Theorem 3.7. *Let X be a projective variety with homogeneous coordinate ring $\mathbf{k}[X]$.*

(a) *$\mathcal{O}(X) = \mathbf{k}$, i.e. the only globally regular functions on X are constant functions.*

Part (a) is related to Liouville's theorem in complex analysis that the only bounded entire functions on $\mathbb{C}$ are constant.

Theorem 3.8. *There is an equivalence of categories given by $X \to \mathbf{k}(X), \phi \to \phi^*$ which is arrow-reversing between quasi-projective varieties and rational maps on finitely generated field extensions of $\mathbf{k}$.*

Some examples considered:

- Veronese Map

- Segre Map
- Grassmannian

Definition 3.9 (Birational isomorphism)**.** Let $\phi : X \to Y$ be a rational map such that $\phi : U \subset X \to Y$ is regular on U, then if ϕ^{-1} map exists and is rational, then we say X is birationally isomorphic to Y (or simply X is birational to Y).

Theorem 3.10. *Let X, Y be quasi-projective varieties. Then there is a bijection between dominant rational maps $\phi : X \dashrightarrow Y$ and $\mathbf{k}$-algebra homomorphisms $\phi^* : \mathbf{k}(Y) \to \mathbf{k}(X)$.*

4. Degree and Hilbert Functions

Let $X \subset \mathbb{P}^n$ be a projective variety. Let $\dim(X) = n$.
A linear function ℓ on $\mathbb{A}^{r+1}$ defines a hyperplane $V(\ell)$ in $\mathbb{P}^r$.

Theorem 4.1 (Degree of a projective variety)**.** *Let $X \subset \mathbb{P}^r$ be an irreducible projective subvariety of dimension n. Let $H_1, \dots, H_n$ be general hyperplanes in $\mathbb{P}^r$. Then the number of intersections*

$$|H_1 \cap \cdots \cap H_n \cap X|$$

is finite and independent of the choice of the H_i. More precisely, there exists a non-empty Zariski open subset $\mathbf{U}$ of the product of dual vector spaces $(\mathbf{k}^{r+1})^* \times \cdots \times (\mathbf{k}^{r+1})^*$ such that, if $H_i = V(\ell_i)$ denote the hyperplane defined by ℓ_i then for any $(\ell_1, \dots, \ell_n) \in \mathbf{U}$ the number of points in $H_1 \cap \cdots \cap H_n \cap X$ is finite and independent of the choice of the $(\ell_1, \dots, \ell_n) \in \mathbf{U}$.

Definition 4.2 (Degree of a projective variety)**.** For general hyperplanes $H_1, \dots, H_n \subset \mathbb{P}^r$, the number $|H_1 \cap \cdots \cap H_n \cap X|$ is called the *degree of* X.

Few examples considered: the degree of the image of the Veronese map (also called the rational normal curve), the degree of a hypersurface.

Definition 4.3 (Hilbert Function)**.** We define the Hilbert function $H_X : \mathbb{N} \to \mathbb{N}$ defined on the natural numbers as follows:

$$H_X(m) = \dim_{\mathbf{k}} \mathbf{k}[X]_m$$

where $\mathbf{k}[X]_m$ denotes the homogenous elements of degree m in the graded ring $\mathbf{k}[X]_m$. More generally, if $M = \oplus_{m \geq 0} M_m$ is a graded $\mathbf{k}[x_0, \dots, x_r]$-module then the Hilbert function H_M is defined by $H_M(m) = \dim_{\mathbf{k}} M_k$.

Theorem 4.4 (Hilbert-Serre)**.** *Let M be a graded $\mathbf{k}[x_0, \dots, x_r]$-module, $M = \bigoplus_{m \geq 0} M_m$. Suppose M is finitely generated. Then there exists a (unique) polynomial $P_M(m)$ with rational coefficients such that for any sufficiently large integer $m > 0$ we have*

$$H_M(m) = P_M(m).$$

The polynomial P_M is called the Hilbert polynomial of the module M.

Definition 4.5 (Hilbert polynomial of a projective variety)**.** Let $X \subset \mathbb{P}^r$ be a projective subvariety. The *Hilbert polynomial P_X of X* is the Hilbert polynomial of its homogeneous coordinate ring $\mathbf{k}[X]$ regarded as a $\mathbf{k}[x_0, \dots, x_r]$-module.

Theorem 4.6 (Hilbert's theorem on degree of a projective variety)**.** *Let $X \subset \mathbb{P}^r$ be a projective subvariety of dimension n. Then:*

(a) *the Hilbert polynomial $P_X(m)$ is a polynomial of degree n.*
(b) *The leading coefficient of P_X is equal to degree of X divided by $n!$.*

5. Bernstein-Kushnirenko theorem

Theorem 5.1 (Kushnirenko)**.** *Let $A \subset \mathbb{Z}^n$ be a finite set. Define $L_A = \{f | f = \sum_{\alpha \in A} c_\alpha x^\alpha\} \subset \mathbb{C}[x_1, \dots, x_n]$. Then the following is true: If $f_1, \dots, f_n$ are general elements in L_A then*

$$N = |\{x \in (\mathbb{C}^*)^n \mid f_1(x) = \cdots = f_n(x) = 0\}|$$

is independent of choice of f_i and

$$N = n!\mathrm{vol}(\Delta)$$

where $\Delta = \mathrm{conv}(A)$.

Let $A = \{\alpha_1, \dots, \alpha_r\} \subset \mathbb{Z}^n$ be a finite set of integral points. Consider the morphism $\Phi_A : (\mathbf{k}^*)^n \to \mathbb{P}^{r-1}$ given by

$$\Phi_A(x) = (x^{\alpha_1} : \cdots : x^{\alpha_r})$$

where we use shorthand notation $x^\alpha = x_1^{a_1} \cdots x_n^{a_n}$ for $x = (x_1, \dots, x_n)$ and $\alpha = (a_1, \dots, a_n)$. Let Y_A denote the closure of the image of the map Φ_A. Then the Bernstein-Kushnirenko theorem alternatively can be stated as:

Theorem 5.2 (Alternative statement of Kushnirenko theorem)**.** *Suppose Φ_A is an embedding (which means that the differences of elements of A generates $\mathbb{Z}^n$). Then the degree of Y_A is $n!\mathrm{vol}(\Delta)$ where $\Delta = \mathrm{conv}(A)$.*

One shows that if $\Delta_1, \Delta_2 \subset \mathbb{R}^n$ are convex bodies (i.e. convex compact subsets) then

$$\Delta_1 + \Delta_2 = \{a + b \mid a \in \Delta_1, b \in \Delta_2\}$$

is also a convex body. $\Delta_1 + \Delta_2$ is called the *Minkowski sum of the convex bodies Δ_1 and Δ_2*. Similarly one can multiply a convex body with a positive real number, i.e. if $c > 0$ and $\Delta \subset \mathbb{R}^n$ then

$$c\Delta = \{ca \mid a \in \Delta\}$$

is a convex body.

Theorem 5.3 (Minkowski)**.** *Let $\Delta_1, \dots, \Delta_r \subset \mathbb{R}^n$ be a finite collection of convex bodies, then the function*

$$f(c_1, \dots, c_r) = \mathrm{vol}(c_1\Delta_1 + \cdots + c_r\Delta_r)$$

is a homogeneous polynomial of degree n in the c_i.

Given a homogeneous polynomial F of degree n on a vector space one can uniquely find an n-linear function B on the vector space such that $F(x) = B(x, \dots, x)$ for any vector x. The n-linear function B is called the *polarization of F*. For example if F is homogeneous of degree 2 then

$$B(x, y) = \frac{F(x + y) - F(x) - F(y)}{2}.$$

This comes from the kindergarten identity $xy = ((x + y)^2 - x^2 - y^2)/2$.

From the above it follows that one can uniquely define a function V on the n-tuples of convex bodies in $\mathbb{R}^n$ with the following properties:

(i) For any convex bodies $\Delta_1, \ldots, \Delta_n \subset \mathbb{R}^n$, $V(\Delta_1, \ldots, \Delta_n) \geq 0$.
(ii) V is symmetric, i.e. does not change under permuting the arguments.
(iii) V is multi-linear, i.e. it is linear in each argument. Linearity in the first argument means the following: let $c, c' > 0$ and $\Delta_1, \Delta'_1, \Delta_2, \ldots, \Delta_n \subset \mathbb{R}^n$ be convex bodies, then

$$V(c\Delta_1 + c'\Delta'_1, \Delta_2, \ldots, \Delta_n) = cV(\Delta_1, \Delta_2, \ldots, \Delta_n) + c'V(\Delta'_1, \Delta_2, \ldots, \Delta_n).$$

(iv) Finally, if $\Delta \subset \mathbb{R}^n$ is a convex body then

$$V(\Delta, \ldots, \Delta) = \mathrm{vol}(\Delta).$$

Definition 5.4. The above n-linear map V is called the *mixed volume of convex bodies.*

Theorem 5.5 (Bernstein-Kushnirenko)**.** *Let $A_1, \ldots, A_n$ be finite subsets of $\mathbb{Z}^n$. Let $L_{A_1}, \ldots, L_{A_n}$ be the corresponding subspaces of Laurent polynomials. Then if $f_1, \ldots, f_n$ are general Laurent polynomials with $f_i \in L_{A_i}$ then the number of solutions*

$$|\{x \in (\mathbb{C}^*)^n \mid f_1(x) = \cdots = f_n(x) = 0\}|$$

is independent of the choice of f_i and is equal to $n!V(\Delta_1, \ldots, \Delta_n)$ where Δ_i denotes the convex hull of the finite set A_i, and V is the mixed volume of convex bodies.

6. Degree as volume

In this section we assume familiarity with manifolds and differential forms. Let us take the ground field to be $\mathbf{k} = \mathbb{C}$, the field of complex numbers. Consider the projective space $\mathbb{P}^n$. It is the quotient of $\mathbb{A}^{n+1}$ by the action of $\mathbb{C}^*$. Alternatively we can consider $\mathbb{P}^n$ as the quotient of the $(2n + 1)$-sphere by the action of the circle S^1. The sphere has a natural metric induced by its embedding in the affine space $\mathbb{A}^{n+1} \setminus \{0\}$ which is equipped with Euclidean metric. The action of S^1 preserves the natural metric on $(2n + 1)$-sphere and hence the projective space $\mathbb{P}^n$ inherits a metric.

Definition 6.1 (Fubini-Study metric)**.** The above (Riemannian) metric on the projective space $\mathbb{P}^n$ is called the *Fubini-Study* metric. It gives a volume form (alternatively measure) on $\mathbb{P}^n$ which sometimes is called the *symplective volume* on $\mathbb{P}^n$.

Theorem 6.2 (Kähler form)**.** (a) *There exists a closed 2-form ω on the projective space $\mathbb{P}^n$ such that ω^n is the Fubini-Study volume form.*

(b) *The Poincare dual of the De Rham cohomology class represented by ω is the class of hyperplanes in $\mathbb{P}^n$ (all hyperplanes in $\mathbb{P}^n$ are homologous and represent a class in $H_2(\mathbb{P}^n, \mathbb{Z})$.*

From the Poincare duality one shows the following:

Corollary 6.3 (Degree as volume)**.** *Let $X \subset \mathbb{P}^n$ be a projective subvariety of dimension d. Then the volume of X with respect to the Fubinu-Study metric, i.e.*

$$\mathrm{vol}(X) = \int_X \omega^d$$

coincides with degree of X.

Remark 6.4. The Bernstein-Kushnirenko theorem on the number of solutions of a general system of Laurent polynomials can be proved using Theorem 6.3. The proof relies on the notion of a *moment map* from symplectic geometry and Hamiltonian group actions.

Remark 6.5. Theorem 6.3 is related to the well-known Crofton formula in integral geometry which relates the length of a plane curve with the average number of intersections points of the curve with lines in the plane. We can parametrize a general line ℓ in $\mathbb{R}^2$ by the direction ϕ it points and its distance p from the origin.

Theorem 6.6 (Crofton formula)**.** *Let γ be a curve in the real plane $\mathbb{R}^2$ with finite length. For each directed line $\ell = \ell(\phi, p)$ let $n_\gamma(\phi, p)$ denote the number of intersections of γ and ℓ. Then the length of γ is equal to:*

$$\frac{1}{4} \int \int n_\gamma(\phi, p) d\phi dp.$$

Note that we are over real numbers and also in general the curve γ is not necessarily algebraic, thus $n_\gamma(\phi, \theta)$ is not the same for a general line line ℓ. See: http://merganser.math.gvsu.edu/david/reed03/projects/weyhaupt/project.html

7. Non-singular varieties

For each polynomial $F(x_1, \ldots, x_n) \in \mathbf{k}[x_1, \ldots, x_n]$ and $p \in \mathbb{A}^n$ let differential dF_p denote the linear function

$$dF_p(v) = \sum_{i=1}^n \frac{\partial F}{\partial x_i}(p) v_i,$$

for $v = (v_1, \ldots, v_n) \in \mathbb{A}^n$.

Let $X = V(I)$ be an affine variety with ideal I.

Definition 7.1 (Tangent space)**.** The tangent space T_pX at a point $p \in X$ is the linear space

$$T_pX = \{v \mid dF_p(v) = 0 \ \forall F \in I\}.$$

It can be verified that if $\{f_1, \ldots, f_r\}$ is a set of generators for I then

$$T_pX = \{v \mid d{f_1}_p(v) = \ldots = d{f_r}_p(v) = 0\}.$$

Thus $\dim_{\mathbf{k}}(T_pX) \geq n - r$.

Definition 7.2 (Non-singular point)**.** Let $X \subset \mathbb{A}^n$ be an affine variety of dimension d. A point $p \in X$ is non-singular if dimension of the tangent space T_pX, as a vector space over $\mathbf{k}$, is d. Equivalently, if $\{f_1, \dots, f_r\}$ is a set of generators for I, $p \in X$ is non-singular if and only if the $r \times n$ matrix of partial derivatives $[\partial f_i/\partial x_j(p)]$ has rank $n - d$.

For a quasi-projective variety $X \subset \mathbb{P}^n$ and $p \in X$ we take an affine open subset U containing p. Then we say p is a non-singular point of X if and only if it is a non-singular point of U.

Let $X = V(I) \subset \mathbf{k}^n$ and assume that the origin O lies on X. Let I_ℓ be the ideal generated by the linear terms f_ℓ of $f \in I$. By definition, $T_OX = V(I_\ell)$. Let $A_\ell = \mathbf{k}[x_1, \dots, x_n]/I_\ell$, and let $\mathfrak{m}$ be the maximal ideal of O in $\mathbf{k}[X]$, thus $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$.

Theorem 7.3 (Intrinsic definition of tangent space)**.** *There are canonical isomorphisms*

$$\mathrm{Hom}_{\mathbf{k}-\mathrm{linear}}(\mathfrak{m}/\mathfrak{m}^2, \mathbf{k}) \cong \mathrm{Hom}_{\mathbf{k}-\mathrm{algebra}}(A_\ell, \mathbf{k}) \cong T_oX.$$

Thus, for any $p \in X$ with maximal ideal $\mathfrak{m}_p \in \mathbf{k}[X]$, the dual vector space T_pX can be identified with the quotient $\mathfrak{m}_p/\mathfrak{m}_p^2$ (regarded as a vector space over $\mathbf{k} = \mathbf{k}[X]/\mathfrak{m}_p$).

Lemma 7.4 (Nakayama's lemma)**.** *Let A be a local ring with maximal ideal $\mathfrak{m}$, and let M be a finitely generated A-module. (a) If $M = \mathfrak{m}M$,then $M = \{0\}$. (b) If N is a submodule of M such that $M = N + \mathfrak{m}M$, then $M = N$.*

Corollary 7.5. *The elements $a_1, \dots, a_n$ of $\mathfrak{m}$ generate $\mathfrak{m}$ as an ideal if and only if their residues modulo $\mathfrak{m}^2$ generate $\mathfrak{m}/\mathfrak{m}^2$ as a vector space over $\mathbf{k}$. In particular, the minimum number of generators for the maximal ideal is equal to the dimension of the vector space $\mathfrak{m}/\mathfrak{m}^2$.*

Definition 7.6 (Regular local ring)**.** A local ring A is *regular* if the maximal ideal $\mathfrak{m}$ of A can be generated by d elements where d is the Krull dimension of A.

Corollary 7.7. *Let X be a variety of dimension d. A point $p \in X$ is non-singular if and only if the local ring $\mathcal{O}_{p,X}$ is regular, i.e. if $\mathfrak{m}_p$ can be generated by d elements. In particular, if X is a curve, a point $p \in X$ is non-singular if and only if $\mathfrak{m}_p$ is principal.*

Theorem 7.8. *Let X be a quasi-projective variety. Then the set of non-singular points of X is a non-empty Zariski open subset of X.*

There is a weaker notion than non-singularity called normality.

Definition 7.9 (Normal variety)**.** Let X be a variety. X is called *normal* at a point $p \in X$ if the local ring $\mathcal{O}_{p,X}$ is integrally closed. The variety X is *normal* if it is normal at every point.

- One shows that an affine variety X is normal if its coordinate ring $\mathbf{k}[X]$ is integrally closed.

- Given a variety X there exists a unique normal variety $\tilde{X}$ together with a morphism $\phi : \tilde{X} \to X$ which is a birational isomorphism. If X is an affine variety then $\tilde{X}$ is the variety associated to the integral closure of $\mathbf{k}[X]$.
- If X is a curve then $\tilde{X}$ is a non-singular curve.

8. Étale maps and inverse function theorem

Theorem 8.1 (Inverse function theorem from calculus)**.** *Let $F = (F_1, \ldots, F_n) : U \to \mathbb{R}^n$ be a differentiable map where $U \subset \mathbb{R}^n$ is an open subset. Take $p \in U$ and assume that the matrix of partial derivatives $[\partial F_i/\partial x_j(p)]$ is invertible. Then there exists open neighborhoods $p \in V$ and $F(p) \in W$ such that $F : V \to W$ is bijective and $F^{-1} : W \to V$ is also differentiable. (Same statement holds over $\mathbb{C}$.)*

Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be affine varieties and $\phi : X \to Y$ a morphism, $\phi = (\phi_1, \ldots, \phi_m)$ where ϕ_i are polynomials in $\mathbf{k}[x_1, \ldots, x_n]$. For any $p \in X$ (not necessarily non-singular) the differential $d\phi_p$ is the linear map given by the matrix of partial derivatives $[\partial\phi_i/\partial x_j]$. One shows that $d\phi$ maps T_pX to $T_{\phi(p)}Y$. By considering affine charts in projective varieties, we can define differential of a morphism between quasi-projective varieties.

The Inverse Function Theorem does not hold in the category of algebraic varieties and morphisms.

Example 8.2. Consider the map $\phi(x) = x^2$. Then for $x \neq 0$ we have $d\phi(x) \neq 0$ but ϕ is not invertible, as a morphism, in a neighborhood of x because $\phi^{-1}(x) = x^{1/2}$ is not a morphism (algebraic map).

Definition 8.3 (Étale morphism)**.** A morphism $\phi : X \to Y$ is étale at a non-singular $p \in X$ if the differential $d\phi : T_pX \to T_{\phi(x)}Y$ is an isomorphism of vector spaces.

Let p be a nonsingular point on a variety V of dimension d . A local system of parame-ters at p is a family $\{f_1, \ldots, f_d\} \subset \mathcal{O}_{p,X}$ of germs of regular functions at p generating the maximal ideal $\mathfrak{m}_p$. Equivalent conditions: the images of $f_1, \ldots, f_d$ in $\mathfrak{m}_p/\mathfrak{m}_p^2$ generate it as a $\mathbf{k}$-vector space, or $df_{1p}, \ldots, df_{dp}$ is a basis for dual space to T_pX.

Definition 8.4 (Étale neighborhood)**.** Let X be a non-singular variety and let $p \in X$. An *étale neighborhood* of p is a pair $(q, \pi : U \to X)$ with π an étale map from a non-singular variety U to X and q a point of U such that $\pi(q) = p$.

Proposition 8.5. *Let X be a non-singular variety of dimension d and let $p \in X$. There is an open Zariski neighborhood U of p and a map $\pi : U \to \mathbb{A}^d$ realizing (p, U) as an étale neighborhood of $(0, \ldots, 0) \in \mathbb{A}^d$.*

Note the analogy of the above proposition with the definition of a differentiable manifold: every point p on a non-singular variety of dimension d has an open neighborhood that is also a "neighborhood" of the origin in $\mathbb{A}^d$. There is a "topology" on algebraic varieties for which the "open neighborhoods" are étale neighborhoods. Relative to this "topology", any two non-singular varieties are locally isomorphic (this is not true for the Zariski topology). The "topology" is called the *étale topology.*

Grothendieck introduced étale topology to construct his étale cohomology which plays a crucial role in the proof of celebrated Weil conjectures on the number of points on varieties over finite fields.

The proposition below is easy corollary of definition.

Proposition 8.6 (Inverse function theorem for étale topology)**.** *If a morphism of non-singular algebraic varieties $\phi : X \to Y$ is étale at $p \in X$, then there exists a commutative diagram*

$$\begin{array}{ccc} U_p & \longrightarrow & X \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} \\ U_{\phi(p)} & \xrightarrow{\text{étale}} & Y \end{array}$$

with U_p an open neighborhood U of p, $U_{\phi(p)}$ an étale neighborhood of $\phi(p)$ and ϕ' an isomorphism.

One can say that the notions of étale map/neighborhood/topology are designed so that the inverse function theorem (as in the above proposition) holds.

9. Blow-up

Construction of blow up of affine space $\mathbb{A}^n$ at the origin. The strict transform. Theorem of resolution of singularities.

The blowup of the affine space $\mathbb{A}^n$ is the variety $X \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$ defined as follows. Every point in the projective space $\mathbb{P}^{n-1}$ can be represented by a line ℓ in $\mathbb{A}^n$ passing through the origin. Then

$$X = \{(p, [\ell]) \in \mathbb{A}^n \times \mathbb{P}^{n-1} \mid p \in \ell\}.$$

Let us use coordinates $(x_1, \ldots, x_n)$ for points in $\mathbb{A}^n$ and homogeneous coordinates $(y_1 : \cdots : y_n)$ for points in $\mathbb{P}^{n-1}$. One verifies that X can be defined by equations $x_i y_j = x_j y_i$ for all i, j.

We explained geometrically that, when the ground field is $\mathbf{k} = \mathbb{R}$, the blowup of $\mathbb{A}^2$ at the origin looks like the Möbius strip.

Let $\phi : X \to \mathbb{A}^n$ be the morphism which is the restriction of the projection on the first factor $\mathbb{A}^n \times \mathbb{P}^{n-1} \to \mathbb{A}^n$. It is not hard to see that for any $p \neq O$, the inverse image $\phi^{-1}(p)$ consists of a single point. On the other hand, $\phi^{-1}(O)$ is a projective space $\mathbb{P}^{n-1}$. Thus X looks like $\mathbb{A}^n$ where a copy of $\mathbb{P}^{n-1}$ is attached at the origin. More strongly, ϕ gives an isomorphism between $X \setminus \phi^{-1}(O)$ and $\mathbb{A}^n \setminus \{O\}$.

Definition 9.1. Let $Y \subset \mathbb{A}^n$ be a subvariety passing through the origin. Then the strict transform $\tilde{Y}$ of Y is the closure of $\phi^{-1}(Y \setminus \{O\})$ in X. The idea is that if Y is singular at O then $\tilde{Y}$ is either non-singular or has milder singularities.

We discussed the example of singular curve $y^2 = x^2(x-1)$ in $\mathbb{A}^2$ and its strict transform after blowing up $\mathbb{A}^2$ at the origin.

We also discussed briefly the famous *theorem of resolution of singularities*. It was proved by Hironaka for char$\mathbf{k} = 0$. It is unsolved for char$\mathbf{k} = p$ but widely believed to be true.

10. Non-singular curves

Definition 10.1 (Valuation)**.** Let K be a field and Γ a totally ordered abelian group. A *valuation* of K with values in Γ is a map $v : K \setminus \{0\} \to \Gamma$ such that for all $x, y \in K$, $x, y \neq 0$ we have:

(i) $v(xy) = v(x) + v(y)$
(ii) $v(x + y) = \min(v(x), v(y))$

A valuation is *discrete* if its value group Γ is $\mathbb{Z}$.

Example 10.2.
- Let $K = \mathbb{Q}$ and p a prime number. For $0 \neq x \in \mathbb{Q}$ let $v_p(x)$ = number of times x is divisible by p. Then v_p is a valuation with values in $\mathbb{Z}$.
- Let $K = \mathbb{C}(t)$ be the field of rational polynomials in one variable t. Let $a \in \mathbb{C}$. Define $v_a(f)$ = order of zero or pole of f at a. Then v_a is a valuation on $\mathbb{C}(t)$ with values in $\mathbb{Z}$.

More generally, if X is any curve over a field $\mathbf{k}$ and $a \in X$ a non-singular point we can define a valuation v_a on the field of rational functions $K = \mathbf{k}(X)$. We will deal with such valuations below.

Definition 10.3 (Valuation ring)**.** If v is valuation then the set $R = \{x \in K \mid v(x) \geq 0\} \cup \{0\}$ is a subring of K called *valuation ring* of v. The ring R is a local ring with unique maximal ideal $\mathfrak{m} = \{x \in K \mid v(x) > 0\} \cup \{0\}$. The valuation ring of a discrete valuation is called a *discrete valuation ring.*

Theorem 10.4 (Characterization of discrete valuation rings)**.** *Let A be a Noetherian local domain of Krull dimension* 1*, with maximal ideal* $\mathfrak{m}$*. Then the following are equivalent:*

(i) *A is a discrete valuation ring.*
(ii) *A is integrally closed.*
(iii) *A is a regular local ring.*
(iv) *$\mathfrak{m}$ is a principal ideal.*

We discussed example of the curve $y^2 = x^2(x + 1)$ to intuitively justify why the above equivalent conditions fail at the singular point $O = (0, 0)$.

Definition 10.5 (Dedekind domain)**.** A *Dedekind domain* is an integrally closed Noetherian domain of dimension 1 (i.e. every non-zero prime ideal is maximal). By above theorem a local Dedeking domain is just a discrete valuation ring. One shows that a domain is a Dedkind domain if and only if its localization at every maximal ideal is a discrete valuation ring.

Dedekind domains are named after famous German mathematician Richard Dedekind who proved unique factorization for ideals in a Dedkind domain. Dedkind is also credited for introducing the fundamental notion of an ideal in a ring. The main examples of Dedkind domain are ring of integers in a number field and the coordinate ring of an affine algebraic variety.

Theorem 10.6. *Any discrete valuation ring of K is isomorphic to the local ring of a point on some non-singular affine curve.*

Definition 10.7. Let K be a function field of dimension 1, i.e. the field of rational functions on some algebraic curve. We denote the collection of all discrete valuation rings of K by C_K.

C_K can be given a topology and structure of sheaf $\mathcal{O}_(C_K)$ of functions: the topology on C_K is the topology in which closed sets are finite sets (open sets are complement of finite sets). Given an open set $U \subset C_K$ the ring of regular functions $\mathcal{O}(U)$ by definition is the ring:

$$\mathcal{O}(U) = \bigcap_{R \in U} R.$$

Definition 10.8 (Abstract non-singular curve)**.** Any open subset of C_K with the induced sheaf of regular functions is called an *abstract non-singular curve.* One can define morphism/isomorphism between abstract non-singular curves and ordinary curves in the usual way.

We used notions of a discrete valuation ring, Dedkind domain and abstract non-singular curve C_K to show:

Theorem 10.9. *Every non-singular quasi-projective curve is isomorphic to an abstract non-singular curve.*

The following theorem is an important property of morphisms between curves. It fails for higher dimensional varieties. It corresponds to the so-called *Riemann extension theorem* for holomorphic functions (in theory of complex functions).

Theorem 10.10 (Extension of morphisms)**.** *Let X be an abstract non-singular curve (or an ordinary non-singular curve), let $p \in X$ and let $\phi : X \setminus \{p\} \to Y$ be a morphism where Y is a projective variety. Then there exists a unique morphism $\overline{\phi} : X \to Y$ extending ϕ.*

Finally we have:

Theorem 10.11. *The abstract non-singular curve C_K is isomorphic to a non-singular projective curve.*

Corollary 10.12.
- (i) *Every abstract non-singular curve is isomorphic to a quasi-projective curve. Every non-singular quasi-projective curve is isomorphic to an open subset of a non-singular projective curve.*
- (ii) *Every curve is birationally equivalent to a non-singular projective curve.*

Corollary 10.13. *The following three categories are equivalent:*
- (i) *non-singular projective curves and dominant morphisms.*
- (ii) *quasi-projective curves and dominant rational maps.*
- (iii) *function fields of dimension* 1 *over* $\mathbf{k}$ *and* $\mathbf{k}$*-homomorphisms.*

Note that the above does not hold for higher dimensional varieties.

11. Divisors, line bundles and Riemann-Roch theorem

Definition 11.1 (Weil divisor)**.** Let X be a variety. A *prime divisor* on X is a closed irreducible hypersurface. A *Weil divisor* on X is a finite formal combination $\sum_i n_i Y_i$ where the Y_i are prime divisors and $n_i \in \mathbb{Z}$. We denote the (free abelian) group of all Weil divisors on X by $\mathrm{Div}(X)$.

We would like to associate to each non-zero rational function $f \in \mathbf{k}(X)$ a divisor. For this we need to make sense of order of zero/pole of f on a prime divisor Y. For this we need the following.

Definition 11.2 (Regular in codimension 1)**.** A variety X is called *regular in codimension* 1 (or *non-singular in codimension* 1) if for any closed irreducible hypersurface $Y \subset X$ the local ring $\mathcal{O}_{Y,X}$ is a regular local ring (and hence a discrete valuation ring because it has dimension 1).

Note that if X is a curve, the above condition is equivalent to X be non-singular. One shows that every non-singular variety is regular in codimension 1 but the converse is not true.

We will assume that X is regular in codimension 1. Take $0 \neq f \in \mathbf{k}(X)$ and let Y be a prime divisor. Since $\mathcal{O}_{Y,X}$ is a discrete valuation ring it corresponds to a valuation v_Y. One shows that given f there are only finitely many prime divisors Y such that $v_Y(f) \neq 0$.

Definition 11.3 (Divisor of a rational function)**.** Define the divisor of f by

$$(f) = \sum_Y v_Y(f) Y,$$

where the sum is over all the prime divisors of X. By the comment before the definition the sum is finite. A divisor is called *principal* if it is the divisor of a rational function.

One can show the following.

Theorem 11.4. *Let X be a non-singular projective curve. Let $0 \neq f \in \mathbf{k}(X)$. Then*

$$\sum_{p \in X} v_p(f) = 0.$$

That is, the sum of zeros and poles of a rational function on a projective non-singular curve is 0.

For $X = \mathbb{P}^1$ the above is a standard statement in complex function theory.

The degree of a divisor $D = \sum_i n_i p_i$ on a curve is by definition the sum $\sum_i n_i$. By the above theorem the degree of a divisor class on a non-singular projective algebraic curve is well-defined (i.e. is independent of the choice of a representative).

From properties of valuations one verifies that

$$(fg) = (f) + (g).$$

Hence the set of principal divisors is a subgroup which we denote by $\mathrm{Prin}(X)$.

Definition 11.5 (Divisor class group)**.** The divisor class group of a variety X is the quotient $\mathrm{Div}(X)/\mathrm{Prin}(X)$.

Example 11.6. The divisor class group of $\mathbb{P}^n$ is $\mathbb{Z}$. When $\mathbf{k} = \mathbb{C}$ there is a natural isomorphism between $\mathrm{Div}(\mathbb{P}^n)$ and the homology group $H_{2n-2}(\mathbb{P}^n, \mathbb{Z})$. This is a general phenomenon and usually the divisor class group of a non-singular projective variety of (complex) dimension n is isomorphic to $(2n-2)$-th homology group.

A (Weil) divisor D on a variety X is called *locally principal* if for every $p \in X$ there is a neighborhood U containing p and a rational function f such that part of D which lies in U coincides with the principal divisor (f) on U.

Definition 11.7 (Cartier divisor)**.** A *Cartier divisor* is a collection of data $\{(U_i, f_i)\}$ where the U_i are an open cover of X and for each i, f_i is a regular function on U_i. Moreover, for any i, j we require that f_i/f_j is a regular nowhere zero function on $U_i \cap U_j$.

- A Cartier divisor gives rise to a locally principal divisor.
- The collection of Cartier divisors form an abelian group which we denote by $\mathrm{CDiv}(X)$. The principal divisors form a subgroup of $\mathrm{CDiv}(X)$.
- When X is smooth the notions of Cartier divisor and Weil divisor are the same, but in general one can have a Weil divisor which is not locally principal and hence does not correspond to a Cartier divisor.

Definition 11.8 (Invertible sheaf)**.** A sheaf of $\mathcal{O}(X)$-modules on a variety X which is locally isomorphic to $\mathcal{O}(X)$ is called an *invertible sheaf.* The collection of invertible sheaves is a group with respect to tensor product of modules called the *Picard group of* X.

The invertible sheaves correspond exactly to line bundles. More precisely, if X is a non-singular variety over $\mathbf{k} = \mathbb{C}$, an invertible sheaf on X corresponds to the sheaf of sections of a line bundle over X regarded as complex manifold.

Theorem 11.9.
(a) *Each Cartier divisor D gives rise to an invertible sheaf $\mathcal{L}_D$.*
(b) *$D \mapsto \mathcal{L}_D$ induces an isomorphism between* $\mathrm{CDiv}(X)/\mathrm{Prin}(X)$ *and the Picard group* $\mathrm{Pic}(X)$.

When X is non-singular, the Picard group and divisor class group are the same. Let X be a non-singular projective curve. Then every principal divisor has degree 0. Let $\mathrm{CDiv}^0(X)$ denote the subgroup of Cartier divisors of degree 0 and define $\mathrm{Pic}^0(X)$ to be $\mathrm{CDiv}^0(X)/\mathrm{Prin}(X)$.

Definition 11.10 (Jacobian variety)**.** The group $\mathrm{Pic}(X)$ can be given structure of a variety and $\mathrm{Pic}^0(X)$ can be regarded as the connected component of $\mathrm{Pic}(X)$. The irreducible variety $\mathrm{Pic}^0(X)$ is usually called the *Jacobian variety* of the curve X.

Example 11.11 (Elliptic curves)**.** Let X be an elliptic curve, that is a non-singular projective cubic curve (in $\mathbb{P}^2$). Then the Jacobian variety of X is isomorphic to X itself. That is, every divisor on X is equivalent to a divisor of the form $p - O$ where O is the unique point on X at infinity.

Remark 11.12.

- The Jacobian of a variety is example of a projective algebraic group. It is an interesting (but not difficult) theorem which states that a projective algebraic group is abelian. Projective algebraic groups are called *abelian varieties*. They are named after Abel and Jacobi because of their fundamental work on complex functions and the so-called Abel-Jacobi theorem. (The abelian groups are named after Abel because of his fundamental work on insolvability of the quintic equation, I do not know if it is a coincidence that abelian varieties are abelian groups!)
- Let X be a non-singular projective curve over $\mathbb{C}$ (i.e. a Riemann surface). One can show that the dimension of the Jacobian variety of X is equal to the number g of handles of X regarded as a real 2-dimensional surface.

A Weil divisor $D = \sum_i n_i Y_i$ is called *effective* written $D > 0$ if all the $n_i \geq 0$. A Cartier divisor $D = \{(U_i, f_i)\}$ is *effective* if for every i, f_i is a regular function on U_i.

Let D be a Cartier divisor with corresponding invertible sheaf $\mathcal{L}_D$. Suppose $\mathcal{L}_D$ has a global section, i.e. $\mathcal{L}_D(X) \neq \{0\}$. Then the space of global sections $\mathcal{L}_D(X)$ can be identified with the vectors space of rational functions

$$L(D) = \{f \in \mathbf{k}(X) \mid (f) + D > 0\} \cup \{0\}.$$

In other words, if the Cartier divisor D is given by the data $\{(U_i, f_i)\}$ then $f \in L(D)$ if and only if for every i, ff_i is regular on U_i. If D is a Weil divisor $D = \sum_i n_i Y_i$, then $f \in L(D)$ if and only if all the coefficients of $(f) + D$ are non-negative. That is, for every i, the order of zero/pole of f on the irreducible hypersurface Y_i is greater than or equal to $-n_i$.

Theorem 11.13. *If X is a projective variety then $L(D)$ is a finite dimensional vector space over* $\mathbf{k}$.

We denote the dimension of $L(D)$by $\ell(D)$.

Let X be a non-singular projective curve. The celebrated Riemann-Roch theorem gives important information about the dimensions of the spaces $L(D)$. It has been generalized to singular curves, as well as higher dimensional varieties. The higher dimensional version (for non-singular varieties) is usually referred to as Hirzbruch-Riemann-Roch.

Theorem 11.14 (Riemann-Roch)**.** *Let X be a non-singular projective curve. Let K be the canonical divisor of X (that is the divisor corresponding to the line bundle of 1-forms on X). Then:*

$$\ell(D) - \ell(K - D) = \deg(D) + 1 - g.$$

Here g is the genus of the curve X (when $\mathbf{k} = \mathbb{C}$, topologically X is a compact orientable surface and g is just its number of handles).

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