Department of Mathematics

University of Pittsburgh

MATH 0430 (Abstract algebra)

Midterm 1, Fall 2011

Instructor: Kiumars Kaveh

Last Name: **Student Number:**

First Name:

TIME ALLOWED: 1 HOUR AND 20 MINUTES. TOTAL MARKS: 100
NO AIDS ALLOWED. WRITE SOLUTIONS ON THE SPACE PROVIDED.

Question	Mark
1	/10
2	/10
3	/20
4	/15
5	/10
6	/15
7	/10
8	/10
9	/2
TOTAL	/100 + 2

1(a).[5 points] State Cayley's theorem.

Every group G is isomorphic to a subgroup of a permutation group.

1(b).[5 points] Define a homomorphism between two groups.

$\phi : G \longrightarrow G'$ is a homomorphism

if :

$\forall \; x, y \in G \qquad \phi(xy) = \phi(x)\phi(y)$.

2.[10 points] Prove only *one* of the following theorems: $\mathbb{Z}_n \times \mathbb{Z}_m \cong \mathbb{Z}_{nm}$ if $(n, m) = 1$, or every subgroup of a cyclic group is cyclic.

See Thm. 11.5 in the text.

See Thm 6.6 in the text.

3.[20 points] True or false? Justify your answer, that is if true give a short explanation why, if false give a counter-example.

(a) If G_1 and G_2 are abelian groups then $G_1 \times G_2$ is also abelian.

True.

$$(x,y) \in G_1, \quad (x',y') \in G_2 \Rightarrow (x,y)(x',y') = (xx', yy') = (x'x, y'y) \quad \text{because } G_1, G_2 \text{ abelian}$$

$$= (x',y')(x,y)$$

so $G_1 \times G_2$ abelian.

(b) The group $(\mathbb{Z}_{12}^*, \times)$ is a cyclic group. Recall that $\mathbb{Z}_{12}^* = \{x \mid 1 \leq x < 12,\ (x, 12) = 1\}$.

False

$$\mathbb{Z}_{12}^* = \{1, 5, 7, 11\}$$

$5^2 = 1 \pmod{12}$

$7^2 = 1 \pmod{12}$

$11^2 = 1 \pmod{12}$

So no element is a generator.

(c) Every 3-cycle is an even permutation.

True.

$$(a\,b\,c) = \underbrace{(a\,b)(b\,c)}_{2 \text{ transpositions}}$$ hence even permutation.

(d) Every cyclic group is abelian.

True

$G = \langle a \rangle$

$$a^n a^m = a^{n+m} = a^m a^n.$$

bf 4.[15 points] Give examples of the following.

(a) A subgroup of $S_3 \times \mathbb{Z}_4$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Let $H = \{e, (12)\} < S_3$

Then $H \cong \mathbb{Z}_2$

Let $K = \{0, 2\} < \mathbb{Z}_4$

Then $K \cong \mathbb{Z}_2$,

We have $H \times K < S_3 \times \mathbb{Z}_4$

and $H \times K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(b) An abelian group of order 8 where all of its elements (aside from identity) have order 2.

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

5.[10 points] Let G be a group. Take an element a in G. Consider the map $\phi : G \to G$ defined uisng a as:

$$\phi(x) = axa^{-1}.$$

Using definition and elementary properties of a group show that $\phi : G \to G$ is an isomorphism.

Consider the map

$$\psi(x) = a^{-1} x a.$$

Then $\phi(\psi(x)) = \cancel{a}(\cancel{a^{-1}} x \cancel{a}) \cancel{a^{-1}} = x$

Similarly $\psi(\phi(x)) = x$.

So ψ is inverse mapping of ϕ. So

ϕ is an invertible mapping (i.e. one-to-one and onto).

Also ϕ is homomorphism:

$$\phi(xy) = a\,xy\,a^{-1} = ax(a^{-1}a)\,y\,a^{-1}$$
$$= (axa^{-1})(aya^{-1})$$
$$= \phi(x)\,\phi(y).$$

6.[15 points] Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 4 & 2 & 3 & 6 & 5 & 1 \end{pmatrix}$$

and $\tau = (21)(2457)$. Write σ and τ as product of disjoint cycles, also determine if they are odd or even.

$$\sigma = (1\ 7)(2\ 4\ 3)(5\ 6)$$

$$(243) = (24)(43)$$

$$\Rightarrow \sigma = (17)(24)(43)(56)$$

so σ is even.

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 3 & 5 & 7 & 6 & 1 \end{pmatrix}$$

$$\tau = (1\ 2\ 4\ 5\ 7)$$

$$\tau = (12)(24)(45)(57)$$

τ is also even.

7.[10 points] Write a list of all (non-isomorphic) finite abelian groups of order $540 = 2^2 \cdot 3^3 \cdot 5$.

$\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

$\mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

$\mathbb{Z}_4 \times \mathbb{Z}_{27} \times \mathbb{Z}_5$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{27} \times \mathbb{Z}_5$

8.[10 points] Show that every group of order p, where p is a prime number, is cyclic.

Let $e \neq a \in G$ with $|G| = p$.

Consider the subgroup $\langle a \rangle = H$ generated by G.

Then $|H| \mid |G| \implies |H| = 1$ or $|H| = p$.

$|H|$ can not be $\underline{1}$ because then $H = \{e\}$

but $e \neq a \in H$. So $|H| = p \implies H = G$ and

hence G is cyclic generated by a.

9.[2 points] Draw cartoon face of Lagrange after he proved his theorem!