Adjoint Functors, the Tensor Product, and Symmetric Algebras

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1 Introduction

In the original preface for *Categories for the Working Mathematician*, Mac Lane writes

"Adjoint functors arise everywhere," [Mac98, pp. vii].

This insinuates that adjoint functors are common, but do not arise trivially, and whatever they may be, only come about as we unravel facts about the objects we study. In addition, given that Mac Lane claims this quote is a slogan, it seems like we will want to appeal to adjoint functors when we want to know certain things about the objects we are studying.

Broadly, adjoint functors give us a way to translate between categories without having to appeal to equivalence of categories to do so. That is, adjoint functors preserve the structure we want from objects, and disregard the structure we don't need, whereas an equivalence of categories demand as much structure invariance as possible. As we will see, the weaker condition reveals interesting connections between the algebraic objects we have discussed in lecture.

This paper has two main goals. The first is to formally introduce the notion of an adjoint functor. This first goal is achieved in Section 2. The second goal is to explore a connection between commutative R-algebras and R-modules through the perspective of adjoint functors, which will be discussed in Section 3. This discussion will take up most of the paper. For this second goal, we need to introduce the tensor product over R-modules, discussed in Section 3.1, and symmetric algebras, discussed in Section 3.2.

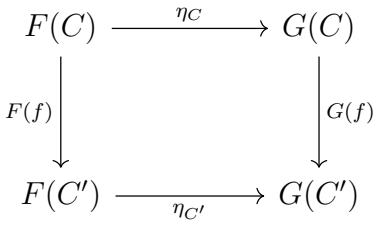
2 Adjunctions

To begin understanding adjunctions, it is worth briefly discussing natural transformations.

2.1 Natural Transformations

Definition 2.1. Let $\mathfrak{C}$ and $\mathfrak{D}$ be categories and $F, G : \mathfrak{C} \to \mathfrak{D}$ be functors. A *natural transformation* $\eta : F \to G$ is a family of maps $\eta_X : F(X) \to G(X)$, for $X \in \mathcal{C}$, in $\mathfrak{D}$ such

that for every $f : C \to C'$, the diagram below is commutative.



We call each η_X *natural* in X. [Lei14, p.28]

The key to understanding natural transformations is understanding that this is a categorical generalization of the idea of a canonical or natural map. For a map to be canonical, we want that the map arises from the structure of the mathematical objects mapping between. Often, it is regarded as the "easiest/most obvious choice," but what does that actually mean? It means that the map commutes with any choice of functor between categories, which is exactly the naturality condition.

To be as concrete as possible, given categories $\mathfrak{C}$ and $\mathfrak{D}$, we can construct the *functor category* $\mathfrak{D}^{\mathfrak{C}}$ whose objects are functors $F : \mathfrak{C} \to \mathfrak{D}$ and whose morphisms are natural transformations. We say a natural transformation η is a *natural isomorphism* if η is an isomorphism in $\mathfrak{D}^{\mathfrak{C}}$.

Example 2.2. As a brief example of a natural transformation, consider `CRing`, the category of commutative rings with ring homomorphisms, and `Mon`, the category of monoids with monoid homomorphisms. Since any commutative ring is also a monoid, given $f \in \mathrm{Hom}_{\texttt{CRing}}(R_1, R_2)$, we can find $\bar{f} \in \mathrm{Hom}_{\texttt{Mon}}(U(R_1), U(R_2))$, where $U : \texttt{CRing} \to \texttt{Mon}$ denotes the forgetful functor. Additionally, for any commutative ring R, $\mathcal{M}_{n\times n}(R)$ is a (multiplicative) monoid, so for any commutative ring homomorphism $R_1 \to R_2$, there is a monoid homomorphism $\mathcal{M}_{n\times n}(R_1) \to \mathcal{M}_{n\times n}(R_2)$. Since we have two different functors between `CRing` and `Mon`, we can ask if there is a natural transformation between them. As it turns out, the familiar $\det_R : \mathcal{M}_{n\times n}(R) \to U(R)$ function is the natural transformation we are looking for. What this tells us is that the definition of determinants is independent of the ring we are working over [Lei14, pp. 29], which should not necessarily be surprising.

2.2 Left and Right Adjoints

Now, we can begin discussing adjoint functors.

Definition 2.3. Given categories $\mathfrak{C}$ and $\mathfrak{D}$ and functors $F : \mathfrak{C} \to \mathfrak{D}$ and $G : \mathfrak{D} \to \mathfrak{C}$, we say F is *left adjoint* to G, and write $F \dashv G$, if, for every $C \in \mathfrak{C}$ and $D \in \mathfrak{D}$, the isomorphism

$$\mathrm{Hom}_{\mathfrak{D}}(F(C), D) \cong \mathrm{Hom}_{\mathfrak{C}}(C, G(D))$$

exists and is natural in both C and D [Lei14, pp. 41]. We also say that G is *right adjoint* to F.

The definition, although quite confusing at first, merely says that a functor is left adjoint to another if morphisms in one category between two objects look exactly like the morphisms in the other category between the original (but translated) objects.[1]

There are a few philosophical takeaways we can draw from how Definition 2.3 is stated. In Definition 2.3, the isomorphism between the Hom-sets tells us that through adjunctions, we can draw a meaningful translation between categories, where problems in each category can be translated between one another. For example, there is an adjunction between the category of sets with n elements and the category of n-dimensional vector spaces [Mac98, p.79]. So a set theorist, who may know very little about linear algebra, and a linear algebraist, who has not brushed up on set theory in quite some time, can have a discussion about the same sorts of objects. The naturality of the isomorphism in Definition 2.3 implies that adjoints are the simplest way to translate between categories because they are canonical. So the adjoints that the set theorist and the linear algebraist come up with will preserve the important structure of the objects they care about, which will greatly benefit their discussion and mutual mathematical understanding.

To get a good sense for adjunctions, let's look at a concrete example.

Example 2.4. For $\mathtt{Gp}$, the category of groups, we can define the *diagonal functor* $\Delta : \mathtt{Gp} \to \mathtt{Gp} \times \mathtt{Gp}$, which takes $G \mapsto (G, G)$.[2] We may ask if the diagonal functor has a left adjoint. In fact, it does, and the left adjoint is familiar. We claim that the free product $* : \mathtt{Gp} \times \mathtt{Gp} \to \mathtt{Gp}$

[1] Of course, there is also the formal aspect that the direction of the morphisms does matter.

[2] In general, the diagonal functor takes $\Delta : \mathfrak{C} \to \mathfrak{C} \times \mathfrak{C}$, where $\mathfrak{C} \times \mathfrak{C}$ is the category where the objects are pairs (G, H) for $G, H \in \mathfrak{C}$ and morphisms are defined coordinate-wise.

is its left adjoint. That is, we claim

$$\mathrm{Hom}_{\mathtt{Gp}\times\mathtt{Gp}}(G_1 \times G_2, \Delta(G)) \cong \mathrm{Hom}_{\mathtt{Gp}}(G_1 * G_2, G).$$

We know that any $f \in \mathrm{Hom}_{\mathtt{Gp}\times\mathtt{Gp}}(G_1 \times G_2, \Delta(G))$ can be decomposed as $f_1 : G \to G_1$ and $f_2 : G \to G_2$. Since the free product is the coproduct in $\mathtt{Gp}$, by definition, any $g \in \mathrm{Hom}_{\mathtt{Gp}}(G_1 * G_2, G)$ can also be decomposed as $g_1 : G \to G_1$ and $g_2 : G \to G_2$. So both Hom-sets are isomorphic to $\mathrm{Hom}_{\mathtt{Gp}}(G_1, G) \times \mathrm{Hom}_{\mathtt{Gp}}(G_2, G)$ and are thus isomorphic to each other. We additionally need this isomorphism to be natural in both $G_1 \times G_2$ and G. If we denote the isomorphism above taking $f \mapsto (f_1, f_2)$ as η, we do indeed have that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathtt{Gp}}(G_1 * G_2, G) & \xrightarrow{\eta_{G_1,G_2,G}} & \mathrm{Hom}_{\mathtt{Gp}\times\mathtt{Gp}}(G_1 \times G_2, \Delta(G)) \\
\Big\downarrow{\scriptstyle \mathrm{Hom}_{\mathtt{Gp}}(f_1 * f_2, p)} & & \Big\downarrow{\scriptstyle \mathrm{Hom}_{\mathtt{Gp}\times\mathtt{Gp}}(f_1 \times f_2, \Delta(p))} \\
\mathrm{Hom}_{\mathtt{Gp}}(H_1 * H_2, H) & \xrightarrow[\eta_{H_1,H_2,H}]{} & \mathrm{Hom}_{\mathtt{Gp}\times\mathtt{Gp}}(H_1 \times H_2, \Delta(H))
\end{array}$$

So $* \dashv \Delta$ [Mac98, pp. 88].

As an important note to this example, notice that we only used the fact that the free product is the coproduct in $\mathtt{Gp}$. In fact, this argument can be generalized to show that the coproduct in a category (if it exists) is left adjoint to the diagonal functor over that category.

We will soon see that in this paper we will be discussing left adjoints quite a bit. But it remains to consider right adjoints as well. The right adjoint of the diagonal functor is the product (if the product exists in the category we are discussing), which follows from analogous reasoning as the coproduct being the left adjoint of the diagonal functor. The important property to exploit is that morphisms into products can be factored into projections onto each factor. It is rather usual for a functor to have both a left and a right adjoint, so functors like with both are rather interesting. Hence, some functors have both a left and a right adjoint, though that is not generally the case. Some functors have just a left adjoint or just a right adjoint. Sometimes they have neither. In any case, if a functor has any adjoint, that is particularly special.

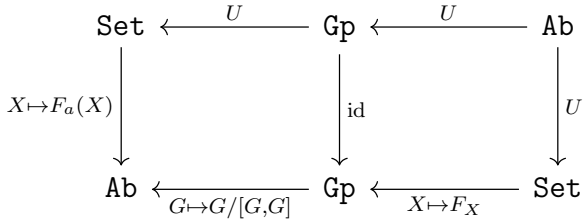
3 An Interesting Example

An example of a type of functor that tends to have a left adjoint is the *forgetful functor*, which takes objects in one category to one of its underlying structures in a different category. [Mac98, p.87] produces an insightful table of forgetful functors and their corresponding left adjoints, which is partially produced and added to in the table below.

Forgetful Functor U	*Left Adjoint to U*
$\texttt{Gp} \to \texttt{Set}$	$X \mapsto F_X$ (free group with generators $x \in X$)
$\texttt{Ab} \to \texttt{Set}$	$X \mapsto F_a(X)$ (free abelian group with generators $x \in X$)
$\texttt{R-mod} \to \texttt{Set}$	$X \mapsto F^R(X)$ (free R-module with basis X)
$\texttt{CommR-alg} \to \texttt{Set}$	$X \mapsto R[X]$ (polynomial ring over R on X)
$\texttt{Field} \to \texttt{Domain}$	$D \mapsto \mathrm{Frac}(D)$ (field of quotients of D)
$\texttt{Ab} \to \texttt{Gp}$	$G \mapsto G/[G,G]$ (factor commutator)

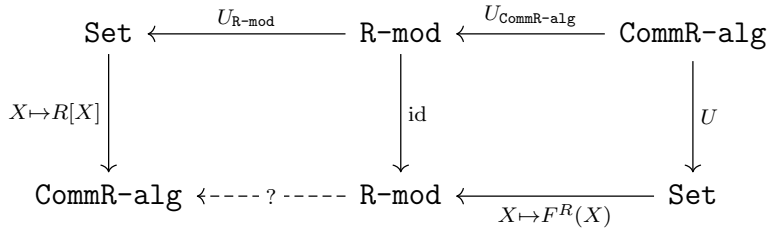
From this table, it seems like left adjoints to forgetful functors between algebraic categories and `Set` often involve a free functor, but left adjoints to forgetful functors between algebraic structure categories are less straightforward. Although the field of quotients and the factor commutator group are fairly simple constructions, they are also not quite trivial. Interesting theory is needed to make these left adjoints precise, namely group theory and field theory.

We also see that we can draw the following diagram of forgetful functors and adjoints:



In fact, the diagram commutes. The table above also implies the following diagram of

forgetful functors and adjoints, but with a notable omission:



Is there a functor taking `R-mod` $\to$ `CommR-alg` that is left adjoint to the forgetful functor U : `CommR-alg` $\to$ `R-mod`? That is, can we canonically construct a commutative R-algebra, given any R-module? The upshot is it is entirely possible, but some preliminary concepts need to be established first.

3.1 The Tensor Product

In order to turn an R-module into an R-algebra, we will need to be able to "piece together" R-modules in a "nice" way. To do this, we need the notion of a tensor product.

Recall the following definition:

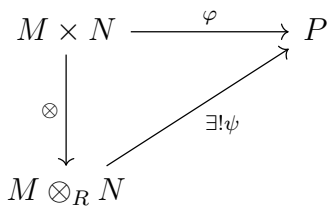
Definition 3.1. For R-modules M, N, P, an *R-bilinear map* is a map $\varphi : M \times N \to P$ such that $n \mapsto \varphi(m, n)$ and $m \mapsto \varphi(m, n)$ are R-module homomorphisms, i.e., φ is R-linear in both factors:

$$\varphi(r_1 m_1 + r_2 m_2, n) = r_1 \varphi(m_1, n) + r_2 \varphi(m_2, n)$$

$$\varphi(m, r_1 n_1 + r_2 n_2) = r_1 \varphi(m, n_1) + r_2 \varphi(m, n_2)$$

for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$, $r_1, r_2 \in R$ [Alu09, pp. 501].

Definition 3.2. Let M, N, P be R-module and $\varphi : M \times N \to P$ an R-bilinear map. We define the *tensor product* of M and N to be the R-module that φ factors uniquely through. We denote this R-module $M \otimes_R N$. We can express this through the following universal property [Alu09, p.501]:



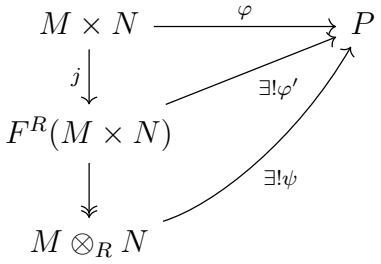
Though, this tensor product is quite mysterious. The universal property does not seem to give any insight into what the tensor product looks like. To demystify the tensor product, we need to construct a compatible $\otimes$ map. We will do this construction the way it is done in [Alu09, pp. 502-503].

Consider a set map $j : M \times N \to F^R(M \times N)$, the free R-module on $M \times N$.[3] In the universal property above, we see that because φ is bilinear, we want $\otimes$ to be R-bilinear. Otherwise, the diagram will not commute. So we need to somehow impose R-bilinearity on j. We can impose R-bilinearity by projecting $F^R(M \times N) \twoheadrightarrow F^R(M \times N)/K$, where

$$K = \langle j(r_1 m_1 + r_2 m_2, n) - r_1 j(m_1, n) - r_2 j(m_2, n), \\ j(m, r_1 n_1 + r_2 n_2) - r_1 j(m, n_1) - r_2 j(m, n_2) : r_1, r_2 \in R \rangle.$$

We claim that the canonical map $M \times N \to F^R(M \times N) \twoheadrightarrow F^R(M \times N)/K$ is $\otimes$ and that $M \otimes_R N \cong F^R(M \times N)/K$.[4] To show this, we just need to show that the specified map satisfies the universal property of the tensor product.

Given an R-bilinear $\varphi : M \times N \to P$ and set map $j : M \times N \to F^R(M \times N)$, by the universal property of free R-modules (Aluffi 167-168), there exists a unique $\varphi' : F^R(M \times N) \to P$ such that $\varphi = \varphi' \circ j$. In addition, $\varphi'(K) = 0$, so by the universal property of quotients, the diagram below is commutative:



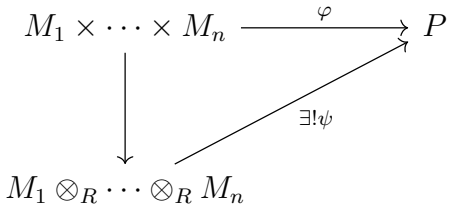
Thus, $M \otimes_R N \cong F^R(M \times N)/K$. So, an element of $M \otimes_R N$ is of the form $m \otimes_R n$, where $m \in M$ and $n \in N$. We call such elements *pure tensors*. The tensor product satisfies many nice properties, including associativity, and by construction, R-bilinearity in both terms. Note that putting the R as a subscript to $\otimes$ does matter. This subscript denotes over what

[3] This R-module really just looks like $R^{\oplus(M \times N)}$.

[4] This is claimed to be an isomorphism rather than equality, which is often show. This is to emphasize that we are defining the tensor product through its universal property.

ring the tensor product is bilinear over. That is, $M \otimes_S N$ is bilinear over R if and only if $S = R$.

By induction, we can tensor arbitrarily-many R-modules. Though, instead of R-bilinearity, we want *R-multilinearity*, or maps $\varphi : M_1 \times \cdots \times M_n \to P$ that are R-linear in all factors [Alu09, pp. 522]. Using induction, it is not hard to show that R-multilinear maps also factor through tensor products:



Example 3.3. We may expect that there is a $\mathbb{Z}$-bilinear map $\varphi : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$. To see what φ looks like, we can factor it through $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z}$. Now the question is, what does this $\mathbb{Z}$-module look like? Notice that $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z}$ is generated by $1 \otimes_{\mathbb{Z}} 1$ but that

$$\begin{aligned} 1 \otimes_{\mathbb{Z}} 1 &= (3-2) \otimes_{\mathbb{Z}} 1 \\ &= (3 \otimes_{\mathbb{Z}} 1) - (2 \otimes_{\mathbb{Z}} 1) \\ &= 3(1 \otimes_{\mathbb{Z}} 1) - (2 \otimes_{\mathbb{Z}} 1) \quad \text{(by bilinearity)} \\ &= (1 \otimes_{\mathbb{Z}} 3) - (2 \otimes_{\mathbb{Z}} 1) \quad \text{(by bilinearity)} \\ &= (1 \otimes_{\mathbb{Z}} 0) - (0 \otimes_{\mathbb{Z}} 1) \\ &= 0. \end{aligned}$$

So, $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \cong 0$.[5] In particular, this means that the only map $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \to P$ is the zero map. Thus, the only $\mathbb{Z}$-bilinear map from $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ is the zero map.

3.2 Symmetric Algebras

To reiterate, the goal of this section is to find a way to turn an R-module into an R-algebra. So, from now on, we will assume that R is commutative.

To begin the construction, we will need some notation. We denote the *nth-fold tensor*

[5] Note that tensoring by zero yields 0. Since the tensor product is bilinear, 0 can be factored out of either factor which kills the tensor.

power of M to be $\mathbb{T}_R^n(M) = M^{\otimes_R n}$ [Alu09, pp. 523]. When $n = 0$, we will say $\mathbb{T}_R^0(M) = R$. Though, this does not define an R-algebra, but rather another R-module. Instead, we will consider the following.

Definition 3.4. We define the *tensor algebra* to be

$$\mathbb{T}_R^*(M) = \bigoplus_{n=0}^{\infty} \mathbb{T}_R^n(M)$$

with multiplication defined by

$$(m_1 \otimes_R \cdots \otimes_R m_i) \cdot (n_1 \otimes_R \cdots \otimes_R n_j) = m_1 \otimes_R \cdots \otimes_R m_i \otimes_R n_1 \otimes_R \cdots \otimes_R n_j,$$

i.e., concatenation of pure tensors [Alu09, pp. 529].

Two natural questions arise. First, why is the tensor algebra an R-algebra? Second, why define multiplication this way?

We will deal with the second question first, which needs some preliminary explanation. The tensor algebra is an example of a *graded algebra*, an algebra S with the decomposition of abelian groups $S = \bigoplus_{n=0}^{\infty} S_n$ such that $S_i \cdot S_j \subseteq S_{i+j}$ and multiplication defined by concatenation [Alu09, pp. 527]. We define a *graded homomorphism* to be a map of rings $\varphi : S \to T$ that preserves grading, i.e., $\varphi(S_i) \subseteq T_i$ [Alu09, pp. 527]. We say a non-zero element of S_i is *homogeneous of degree* i [Alu09, pp. 527]. The exact way to think of these graded algebras is to think of a polynomial algebra over R, where we say that a polynomial composed of variables of the same degree is homogeneous. Notice that the definition of a graded algebra exactly gives us why this multiplication is defined in this way. Given $m_1 \otimes_R \cdots \otimes_R m_i \in \mathbb{T}_R^i(M)$ and $n_1 \otimes_R \cdots \otimes_R n_j \in \mathbb{T}_R^j(M)$, we have that $m_1 \otimes_R \cdots \otimes_R m_i \otimes_R n_1 \otimes_R \cdots \otimes_R n_j \in \mathbb{T}_R^{i+j}(M)$.

Though, in order to even justify the above, we have to say why the tensor algebra is an R-algebra. Thankfully this is not difficult. Since each $\mathbb{T}_R^n(M)$ is an R-module, then $\mathbb{T}_R^*(M)$, the direct sum of these R-modules, is an R-module as well. We also defined a closed multiplication on $\mathbb{T}_R^*(M)$, which is compatible with R-scalar multiplication by the R-multilinearity of the tensor product. Finally, $1_R \in R = \mathbb{T}_R^0(M) \subseteq \mathbb{T}_R^*(M)$, so $\mathbb{T}_R^*(M)$ has both an R-module structure and a ring structure with unity. Since R is assumed to be commutative, then $\mathbb{T}_R^*(M)$ is indeed an R-algebra.

So, we have taken an R-module M and constructed an R-algebra. The key issue is that we have not created a *commutative* R-algebra. To remedy this, we will do the same construction again, accounting for commutativity.

Definition 3.5. We denote the *nth-fold symmetric power of* M to be $\mathbb{S}^n_R(M) = \mathbb{T}^n_R(M)/I$, where I is generated by $(m_1 \otimes_R m_2) - (m_2 \otimes_R m_1)$ for all $m_1, m_2 \in M$. We further define the *symmetric algebra on* M to be

$$\mathbb{S}^*_R(M) = \bigoplus_{n=0}^{\infty} \mathbb{S}^n_R(M),$$

with multiplication defined as in the tensor algebra [Alu09, pp. 529].

For the same reasoning as before, the symmetric algebra is an R-algebra and is also a graded algebra. In fact, we can characterize the symmetric algebra in a slightly nicer way. If we consider the natural map $\varphi : \mathbb{T}^*_R(M) \to \mathbb{S}^*_R(M)$, it is clear that the kernel of φ is I. So, in particular,

$$\mathbb{S}^*_R(M) \cong \frac{\mathbb{T}^*_R(M)}{I}.$$

We seem to have reached our goal.

3.3 The Left Adjoint

It should be clear now that all that is left for us to do is prove the following proposition.

Proposition 3.6. *The left adjoint to the forgetful functor* $U : \texttt{CommR-alg} \to \texttt{R-alg}$ *is the functor* $\mathbb{S}^*_R(-) : \texttt{R-alg} \to \texttt{CommR-alg}$.

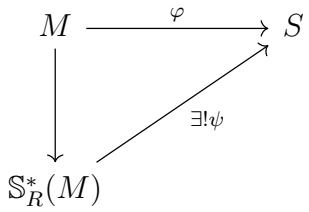
Recall that this means we are claiming that the isomorphism

$$\mathrm{Hom}_{\texttt{CommR-alg}}(\mathbb{S}^*_R(M), S) \cong \mathrm{Hom}_{\texttt{R-mod}}(M, U(S))$$

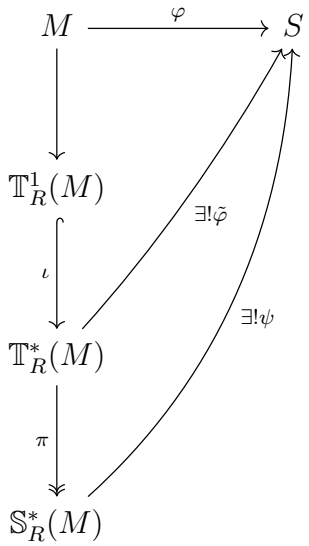
is natural in both $M \in \texttt{R-mod}$ and $S \in \texttt{CommR-alg}$. We will get this isomorphism directly from a nice universal property [Alu09, pp. 531].

Proposition 3.7 (The Universal Property of Symmetric Algebras)**.** *Let* R *be a commutative ring and* M *an* R*-module. Then, for all commutative* R*-algebras* S *and* R*-module homomorphisms* $\varphi : M \to S$*, there exists a unique homomorphism of* R*-algebras* $\psi : \mathbb{S}^*_R(M) \to S$ *such*

that the diagram below commutes.



Proof. Let $\varphi : M \to S$ be an R-module homomorphism. To prove the proposition, we only have to justify the following commutative diagram:



Since $\mathbb{T}_R^1(M) = M$, we can define the map $M \to \mathbb{T}_R^1(M)$ to be the identity map. Then $\iota : \mathbb{T}_R^1(M) \to \mathbb{T}_R^*(M)$ is the natural inclusion, and by the universal property of the direct sum, such unique R-algebra homomorphism $\tilde{\varphi}$ exists as above. Finally, since $\mathbb{S}_R^*(M) \cong \frac{\mathbb{T}_R^*(M)}{I}$, we define the map $\pi : \mathbb{T}_R^*(M) \to \mathbb{S}_R^*(M)$ to be the natural projection, and by the universal property of quotients, we have the unique R-algebra homomorphism ψ exists as above. The entire diagram commutes, so we have proven the claim. □

Now, we can prove the result of this section.

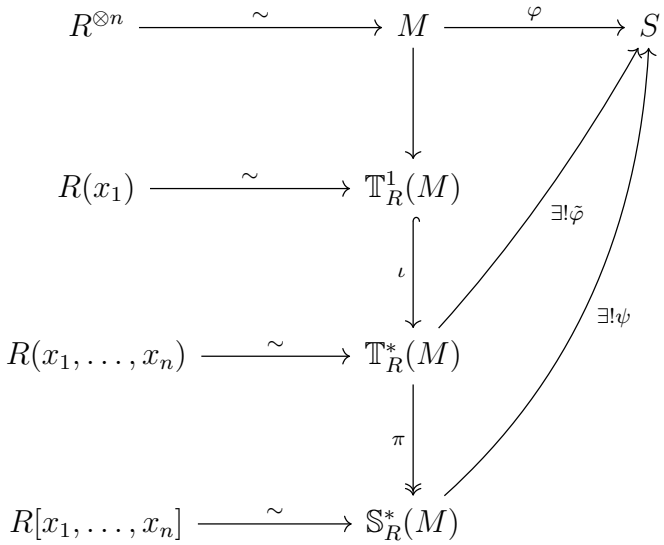
Proof of Proposition 3.6. We want to show that the map

$$\mathrm{Hom}_{\texttt{R-mod}}(M, U(S)) \to \mathrm{Hom}_{\texttt{CommR-alg}}(\mathbb{S}_R^*(M), S),$$

is a natural isomorphism. Take $\varphi \in \mathrm{Hom}_{\texttt{R-mod}}(M, U(S))$. By Proposition 3.7, there exists a unique $\psi \in \mathrm{Hom}_{\texttt{CommR-alg}}(\mathbb{S}_R^*(M), S)$ such that $\varphi \mapsto \psi$, so the map is injective. On the other hand, any $\psi \in \mathrm{Hom}_{\texttt{CommR-alg}}(\mathbb{S}_R^*(M), S)$ is determined by $\psi \in \mathrm{Hom}_{\texttt{R-mod}}(M, U(S))$, so the map is surjective. Finally, the map is a natural isomorphism because of the uniqueness of ψ given φ. Thus, the map is a natural isomorphism, so $\mathbb{S}_R^*(-) \dashv U : \texttt{CommR-alg} \to \texttt{R-alg}$. $\square$

Notice that we have implicitly provided a second construction. In Proposition 3.7, we proved a universal property for the tensor algebra, which through the same reasoning as Proposition 3.6, proves that the functor $\mathbb{T}_R^*(-) \dashv U : \texttt{R-alg} \to \texttt{R-mod}$. This would be interesting enough, but the commutative R-algebra case is particularly appealing.[6]

Finally, it is worth noting the case when M is free, i.e., $M \cong R^{\oplus n}$. In this case, we have the following commutative diagram:



This gives us a nice way to see what exactly is happening in this proof in terms of familiar R-algebras.

[6]For example, topics like algebraic geometry and algebraic number theory use commutative algebra, where commutativity is necessary to do the constructions.

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