A Categorical Framework for the Double-Negation Translation and Modal Translation

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In this paper, the double-negation translation and modal translation between classical propositional logic (CPL) and intuitionistic propositional logic (IPL) will be formalized within the framework of lattice theory and category theory in order to understand their lack of an apparent logic-theoretic relationship. First, the translations will be briefly discussed in the context of CPL and IPL. Next, common notions in lattice theory and category theory will be developed in the context of CPL and IPL, including the categories Heytalg and Boolalg. Then, the double-negation translation will be interpreted as a functor from Heytalg $\to$ Boolalg and analyzed as a left adjoint. Finally, we will examine the possible ways we may be able to interpret the modal translation as a functor to show that the double-negation translation and the modal translation do not have a natural category-theoretic relationship, which informs their lack of a natural logic-theoretic relationship.

Before beginning, we will first lay out the context for our discussion. At first, CPL and IPL may seem unintelligible to each other, but there is a way to translate between logics. On the one hand, given sentences in CPL, the *double-negation translation*, which translates φ as $\neg\neg\varphi$, makes CPL intelligible in IPL [6, pp. 164]. On the other hand, given sentences in IPL, the *modal translation*, which translates ψ as $\Box\psi$,[1] makes IPL intelligible in CPL [6, pp. 166]. Although these translations are well-defined and work, they seem to be of different kinds. The double-negation translation forces IPL to look like CPL by applying an operation that makes sense in both IPL and CPL, while the modal translation has to use modal logic to translate. What is the relationship between these translations? If they were inverses to one another, then in particular,

$$\neg\neg(\Box\psi) = \psi \quad \text{and} \quad \Box(\neg\neg\varphi) = \varphi.$$

But this does not seem correct. If we interpret ψ as "one plus one equals two," then $\neg\neg(\Box\psi)$ is "it is not consistent that one plus one does not equal two," or "it is provable that one plus one equals two," which is stronger than our original statement. So at the very least, these translations cannot be two-sided inverses to one another. Since the relationship between these translations is not clear just applying formal logic, we

[1] In this case, we are interpreting $\Box\psi$ as "it is provable that ψ."

will find suitable models for CPL and IPL such that these translations can be studied analytically. We will do this by formalizing the structure of CPL and IPL in the context of categories and lattices.

First, we need to discuss what a lattice is. A *partially ordered set* (or *poset*) is a tuple $\langle P, \leq_P \rangle$, where P is a set and $\leq_P$ is a reflexive, anti-symmetric, and transitive relation. A good example of a poset is $\langle \mathbb{Z}, \leq_{\mathbb{Z}} \rangle$, where $\leq_{\mathbb{Z}}$ is the usual "less-than-or-equal" relation. From now on, we will refer to the tuple $\langle P, \leq_P \rangle$ as P, implicitly including $\leq_P$.

With this notion, we can elevate posets to lattices, which will be the underlying structure of the objects we will study. A *lattice* is a poset, L such that for any $a, b \in L$, $\inf\{a, b\}, \sup\{a, b\} \in L$ [3, pp. 2]. Notice that although upper and lower bounds of any two elements exist in a lattice, this does not mean that in general any two elements can be compared. Although $\mathbb{Z}$ with $\leq_{\mathbb{Z}}$ as "less-than-or-equal" allows us to compare any two elements easily, we may also consider the lattice $\mathcal{P}(X)$, for any set X, equipped with set inclusion as $\leq_{\mathcal{P}(X)}$. In the case $X = \{1, 2, 3\}$, it is not clear how to directly compare $\{1, 2\}$ and $\{1, 3\}$.[2]

However, the set-theoretic definition of lattices will not suffice. We additionally want to import logic operations on our lattice, so we need to elevate the set-theoretic definition of lattices into an algebraic-theoretic definition. Since a set of any two elements of a lattice admits an infimum and supremum in L, we can define the following operations of any lattice:

$$a \wedge_L b := \inf\{a, b\} \qquad a \vee_L b := \sup\{a, b\}$$

We call these *meets* and *joins*, respectively [3, pp. 4]. It is easy to verify that meets and joins are idempotent,[3] associative, and commutative, and that, by induction, a lattice is closed under finitely-many meets and joins. We also see that, if $a \leq_L b$,

$$a \wedge_L (a \vee_L b) = a \vee_L (a \wedge_L b) = a.$$

[2] In this case, $\sup\{\{1, 2\}, \{1, 3\}\} = X$.

[3] Idempotency is the property that $a \wedge_L a = a$ and $a \vee_L a = a$ [3, pp. 5].

These properties of $\wedge_L$ and $\vee_L$ give us that

$$a \leq_L b \text{ iff } a \wedge_L b = a \qquad\qquad a \leq_L b \text{ iff } a \vee_L b = b$$

[3, pp. 5]. This tells us that we can describe lattices $\langle L, \leq_L \rangle$ equivalently as algebras $\langle L, \wedge_L, \vee_L \rangle$ [3, pp. 6]. Often, we see that lattices do not have a maximal or minimal element.[4] However, if there is a maximal and minimal element of a lattice, we say that the lattice L is *bounded*. We denote the maximal element of a bounded L to be 1 and the minimal element of L to be 0. So, we have that 1 is the meet-identity and 0 is the join-identity, i.e. $a \wedge_L 1 = 1 \wedge_L a = a$ and $a \vee_L 0 = 0 \vee_L a = a$.

Before continuing, it is important to understand what a category is, as it will be vital to our discussion that we can interpret lattice theory as an instance of category theory. Informally, a *category* $\mathcal{C}$ consists of a collection of objects and arrows between these objects. Formally, we refer to a category $\mathcal{C}$ as consisting of a class of objects and a class[5] of morphisms (or arrows) between these objects, such that there is an identity morphism, and the composition of any two morphisms is also a morphism in the $\mathcal{C}$. In addition, these morphisms preserve the structure of the objects in the category, whatever that may mean for the category we are working with. A morphism is an *isomorphism* if it is bijective, which we denote $C \cong D$. The first example of a category is the category `Set`, whose objects are sets and the morphisms of which are functions between sets. Clearly these morphisms are well-defined, since the composition of functions between sets is still a function between sets and there is an identity function.

If we now wanted to describe the collection of all lattices as a category, we would need to understand what morphisms between lattices look like. Given lattices L_1, L_2, a *lattice homomorphism* [3, pp. 19] is a function $\varphi : L_1 \to L_2$ such that φ is a $\wedge$-homomorphism and a $\vee$-homomorphism, i.e.,

$$\varphi(a \wedge_{L_1} b) = \varphi(a) \wedge_{L_2} \varphi(b) \qquad\qquad \varphi(a \vee_{L_1} b) = \varphi(a) \vee_{L_2} \varphi(b),$$

[4] For example, $\mathbb{Z}$.

[5] The word "class" is used in this definition, because, in general, the collection of objects need not be sets, the morphisms between objects need not be functions, and the collection of morphisms need not be sets.

respectively. It is not hard to verify that the identity lattice homomorphism exists and that composition of lattice homomorphisms yields a lattice homomorphism. Thus, we can define the category `Lattice` to be the category whose objects are lattices and morphisms are lattice homomorphisms.

Now that we have defined `Lattice`, it will be easier to discuss the suitable algebraic models for `CPL` and `IPL`.

A *Hetying algebra* is an algebra $\langle H, \wedge_H, \vee_H, \rightarrow_H \rangle$, such that $\langle H, \wedge_H, \vee_H \rangle$ is a bounded lattice and the third operation $\rightarrow_H$ satisfies

$$a \wedge_H x \leq_H b \text{ iff } x \leq_H a \rightarrow_H b$$

for $a, b, x \in H$ [2, pp. 10]. That is,

$$a \rightarrow_H b = \sup\{x \in H : a \wedge_H x \leq_H b\}$$

[2, pp. 10]. We define `Heytalg` to be the category with objects as Heyting algebras and morphisms as lattice homomorphisms that preserve $\rightarrow$ and send $v(0) = 0$, $v(1) = 1$. We call such morphisms *Heyting algebra homomorphisms* [1]. Because the objects and morphisms of `Heytalg` are a subset of the objects and morphisms of `Lattice`, we say that `Heytalg` is a *subcategory* of `Lattice`.

The claim is that we can model `IPL` using Heyting algebras. We define a *valuation* v on a Heyting algebra H to be a function that sends propositional letters to elements of H and the logical connectives to the corresponding connectives of H, such that $v(\bot) = 0$ [2, pp. 11]. We say that H *validates* φ, written $H \models \varphi$, if, for all valuations v, $v(\varphi) = 1$ [2, pp. 11], which can be denoted $H \models^v \varphi$. Using valuations, it is fairly easy to see how a sentence like $\varphi \wedge (\psi \vee \xi)$ could be interpreted in a Heyting algebra. It is less obvious from the definition of $\rightarrow_H$ to see how $\varphi \rightarrow \psi$ is interpreted as $v(\varphi) \rightarrow_H v(\psi)$, and how negation can be interpreted as well.

Whether you subscribe to material conditionalism or a different indicative conditional philosophy, you would at least want that if φ and ψ are true, then $\varphi \rightarrow \psi$ is

true. We see from the definition of $\to_H$ that if $H \models \varphi$ and $H \models \psi$, then

$$v(\varphi) \to_H v(\psi) = 1 \to_H 1 = 1.$$

By definition, $v(\varphi \to \psi) = v(\varphi) \to_H v(\psi)$, so $H \models \varphi \to \psi$. Unfortunately for the material conditionalist, we also see that following:

- If $H \not\models \varphi$ and $H \models \psi$, then $v(\varphi) = a \in H \setminus \{1\}$ and

  $$v(\varphi) \to_H v(\psi) = a \to_H 1 = a \neq 1.$$

  So in this case, $H \not\models \varphi \to \psi$.

- Similarly, if $H \models \varphi$ and $H \not\models \psi$, then $v(\psi) = b \in H \setminus \{1\}$ and

  $$v(\varphi) \to_H v(\psi) = 1 \to_H b = b \neq 1.$$

  So in this case as well, $H \not\models \varphi \to \psi$.

In fact, this definition validates all the axioms of `IPL` [2, pp. 4], so implication is defined how we would like.

We finally define $\neg a := a \to_H 0$ for $a \in H$ [2, pp. 10]. Does this actually work well? Using the definition for $\to_H$, we see that $\neg a$ is the maximal element x of H such that $a \wedge_H x = 0$.[6] We also see that if $a = 0$, then $\neg a = 1$. But what if $0 \leq_H a$? Does this imply that $\neg a = 0$? Take the lattice $\mathcal{P}(\{1, 2\})$ with $\leq_{\mathcal{P}(\{1,2\})}$ as set-inclusion. We define $\wedge_{\mathcal{P}(\{1,2\})}$ to be set-intersection, $\vee_{\mathcal{P}(\{1,2\})}$ to be set-union, and $\emptyset$ to be the minimal element. We see that $\emptyset \subseteq \{1\}$ but

$$\neg\{1\} = \{1\} \to_{\mathcal{P}(\{1,2\})} \emptyset = \sup\{x \in \mathcal{P}(\{1, 2\}) : \{1\} \cap x \subseteq \emptyset\} = \{2\} \neq \emptyset,$$

so it is not always the case that $0 \leq_H a$ implies $\neg a = 0$. This should not disappoint us, since this means, in general, $H \not\models \varphi \vee \neg\varphi$, and Heyting algebras model `IPL`.

Similarly, we can define a model for `CPL` using Boolean algebras. Abstractly, a *Boolean algebra* B is a Heyting algebra, such that every element $b \in B$ has a comple-

[6] Since 0 is the minimal element of H, the condition $a \wedge_H x \leq_H 0$ is equivalent to $a \wedge_H x = 0$.

ment [3, pp. 58], i.e., for all $b \in B$, there exists $\neg b \in B$ such that

$$b \vee_B \neg b = 1 \qquad \text{(Law of Excluded Middle).}^{7}$$

Since we can define Boolean algebras as Heyting algebras and therefore lattices, we can define the category `Boolalg` to be the category whose objects are Boolean algebras and whose morphisms are Heyting algebra homomorphisms. We call such morphisms *Boolean algebra homomorphisms*.[8] Notice that `Boolalg` is a subcategory of both `Lattice` and `Heytalg`. In addition, `CPL` is derived from `IPL` by adding the law of excluded middle, and Boolean algebras are Heyting algebras with the law of excluded middle, so we can model `CPL` using Boolean algebras in the same way we modeled `IPL` with Heyting algebras.

Since we have defined `Heytalg` and `Boolalg`, we will now briefly develop some more category theory in order to construct and analyze the double-negation translation as a functor.

Within a category $\mathcal{C}$, we use morphisms to take objects of $\mathcal{C}$ to other objects of $\mathcal{C}$. With this same perspective, we can think of translations between categories as morphisms between different categories, which we call *functors* [5, pp. 13]. In our definition of a category, it was noted that morphisms preserve the structure of the objects in the category we are working with. So, a functor should preserve the structure of the categories, and since categories are defined by objects and morphisms, a functor should be defined by how it acts on objects and morphisms. In particular, a functor $\mathfrak{F}$ between categories $\mathcal{C}$ and $\mathcal{D}$ take objects $C \in \mathcal{C}$ to objects $D \in \mathcal{D}$ and morphisms in $\mathcal{C}$ to morphisms in $\mathcal{D}$, such that $\mathfrak{F}$ preserves composition and such that the identity morphism in $\mathcal{C}$ is sent to the identity functor in $\mathcal{D}$ [5, pp. 13].

Since `Boolalg` is a subcategory of `Heytalg`, the most "straightforward" functor would be the *forgetful functor* U : `Boolalg` $\to$ `Heytalg`. Forgetful functors are a general type of functor that takes a category to an underlying categorical structure,

[7] Note that since B satisfies the law of excluded middle, B satisfies the law $b \wedge_B \neg b = 0$, which is the secondary condition of the definition of a Boolean algebra in [3, pp. 57].

[8] We don't need to note that Boolean algebra homomorphisms preserve the law of excluded middle because they already preserve $\vee$ and $\to$ (and hence $\neg$).

"forgetting" some of the structure of the original category. In this case, U takes a Boolean algebra to its underlying Heyting algebra structure by forgetting that we need the law of excluded middle to hold. Since `Heytalg` and `Boolalg` have the same morphisms, the forgetful functor does nothing to the morphisms. In the other direction, consider the functor $\neg\neg(-) : \texttt{Heytalg} \to \texttt{Boolalg}$ taking a Heyting algebra H to the Boolean algebra $\neg\neg H$, where

$$\neg\neg H = \{h \in H : \neg\neg h = h\}.$$

Indeed, double-negation elimination is equivalent to the law of excluded middle [2, pp. 4], so $\neg\neg H$ is a Heyting algebra satisfying the law of excluded middle, i.e., a Boolean algebra. Though, it is non-trivial that φ is true in `CPL` if and only if $\neg\neg\varphi$ is true in `IPL`,[9] so the lattice perspective helps makes this relationship clearer.

We see that as U forgets double-negation elimination on a Boolean algebra, $\neg\neg(-)$ restricts a Heyting algebra to its Boolean sublattice, so we would be interested in whether U and $\neg\neg(-)$ are inverses. Notice that because a Heyting algebra H does not generally satisfy the law of excluded middle, $\neg\neg H = H$ if and only if H is a Boolean algebra. So, for $H \in \texttt{Heytalg}$, $U(\neg\neg(H)) \subseteq H$, where $U(\neg\neg(H)) = H$ if and only if $H \in \texttt{Boolalg}$. Thus these functors cannot be inverses. What then is the relationship between these two functors?

We first adopt the notation $\mathrm{Hom}_{\mathcal{C}}(A, B)$ to be the set of all morphisms in $\mathcal{C}$ that takes A to B. For categories $\mathcal{C}$ and $\mathcal{D}$, the functor $F : \mathcal{C} \to \mathcal{D}$ is *left adjoint* to the functor $G : \mathcal{D} \to \mathcal{C}$ if the map

$$\mathrm{Hom}_{\mathcal{D}}(F(C), D) \to \mathrm{Hom}_{\mathcal{C}}(C, G(D))$$

is a natural isomorphism that is natural in C and D [4, pp. 41]. When we talk about "naturality," we are talking about a categorical concept that generalizes the idea of canonical (or natural) maps of functors [4, pp. 28], but the whole statement means that the set $\mathrm{Hom}_{\mathcal{D}}(F(C), D)$ has the same structure as the set $\mathrm{Hom}_{\mathcal{C}}(C, G(D))$ in such a way that is independent of your choice of C and D. That is, the isomorphism arises

[9] This was proved by Glivenko in 1929 and enhanced by Gödel in 1933 [2, pp. 4].

directly from the structure of the objects.[10]

The correct relationship between $\neg\neg(-)$ and U is that $\neg\neg(-)$ is left adjoint to U. That is, for any $B \in \texttt{Boolalg}$ and $H \in \texttt{Heytalg}$,

$$\mathrm{Hom}_{\texttt{Heytalg}}(U(B), H) \cong \mathrm{Hom}_{\texttt{Boolalg}}(B, \neg\neg(H)).$$

If $f \in \mathrm{Hom}_{\texttt{Boolalg}}(B, \neg\neg(H))$, then f is uniquely determined by its underlying Heyting algebra homomorphism $f \in \mathrm{Hom}_{\texttt{Heytalg}}(U(B), H)$. On the other hand, if $g \in \mathrm{Hom}_{\texttt{Heytalg}}(U(B), H)$, then the Boolean algebra homomorphism $g' \in \mathrm{Hom}_{\texttt{Boolalg}}(B, \neg\neg(H))$ has unique underlying Heyting algebra homomorphism g. Since this restriction to a subcategory and this lifting to a supercategory[11] was canonical, the isomorphism is natural. Thus, we have shown that the functor $\neg\neg(-)$ is left adjoint to the forgetful functor U.

So, we have turned the double-elimination translation into a functor. We also want to do this for the modal translation from `CPL` into `IPL`, where a formula φ in `CPL` is interpreted as $\Box\varphi$ in `IPL`. How can we interpret this as a functor $\Box(-) : \texttt{Boolalg} \to \texttt{Heytalg}$?

There are two arguments available to us to see what $\Box(-)$ looks like and what the relationship between the functors $\neg\neg(-)$ and $\Box(-)$ is. The first would be to explicitly define $\Box(-)$. We would want $\Box(-)$ to take a Boolean algebra B to a Heyting algebra $\Box B$, but it is not clear how we can define the $\Box$ operator on a lattice. The standard way to begin this is to embed a Kripke frame into a Heyting algebra H by defining a topology on H made from chains of $\leq_H$ implications, from which we can define $\Box\varphi$ as an inclusion of certain open sets [2, pp. 17-19]. But defining Kripke frames in this way is problematic to our discussion. Defining $\Box B$ in terms of topology means that we would have to consider B and $\Box B$ as objects in the category `Top` of topological spaces, which is not an algebraic category, rather than an object in `Lattice`,[12] which

[10] From now on in this paper, when something is referred to as "natural," we mean it in this sense.

[11] This was not rigorously defined, but hopefully the idea is clear. A *supercategory* of $\mathcal{C}$ is a category that contains $\mathcal{C}$.

[12] The details of this correspondence between `Boolalg` and `Top`, particularly Stone duality, is itself worthy of an entire paper similar in style to this one. Perhaps in the future it will be given a similar treatment.

is an algebraic category. So if we define $\Box(-)$ in this way, we see that this functor is not compatible as a functor between algebraic categories.

The second argument would be to try to force a relationship between $\neg\neg(-)$ and $\Box(-)$ in the way we would like. That is, we suppose $\Box(-)$ is a functor between algebraic categories and see how it could relate to $\neg\neg(-)$. As noted at the beginning of this paper, $\neg\neg(-)$ and $\Box(-)$ do not seem to be inverses of each other. The last real hope that we could have for $\neg\neg(-)$ and $\Box(-)$ to relate nicely, would be to force $\Box(-)$ to be the forgetful functor U, whose left adjoint is $\neg\neg(-)$. This would tell us that $\neg\neg(-)$ and $\Box(-)$ undo each other's structures in the most natural way possible, which is what we would like. Unfortunately, there is a problem with this. We defined the forgetful functor as "forgetting" the law of excluded middle, meaning we no longer enforced the simplification that $p \vee_B \neg p = 1$. So if we said that $\Box(-)$ was exactly U, then we would be claiming that the $\Box$ operator, however we should define it over algebraic semantics, should serve to forget the law of excluded middle. Instead, we seem to be claiming that $p \vee_B \neg p$ is provable, which is problematic in a general Heyting algebra. In addition, $\Box$ is interpreted as "is provable," meaning $\Box\varphi$ is true if φ follows from true statements. Hence, on a Boolean lattice B, we have to group together chains of $\leq_B$ (in some way) such that the law of excluded middle does not need to hold. This must be done canonically, otherwise the relationship to $\neg\neg(-)$ would not be preserved. In particular, we cannot pick these groups element-by-element and instead have to find a way to do this no matter the Boolean algebra we started with. Once we start defining canonical groupings of elements of a Boolean algebra, we very quickly see that we are trying to define a topology on Boolean algebras, which was not our goal.

So what is left from these two arguments? Perhaps we want translations between intuitionistic and `CPL` to prioritize naturality and therefore reject $\Box(-) : \texttt{Boolalg} \to \texttt{Heytalg}$ as a suitable translation. Then we would likely have to accept $U : \texttt{Boolalg} \to \texttt{Heytalg}$ as the best translation from `CPL` to `IPL`, which does not work either: if we did accept U as the best translation, $p \vee \neg p = 1$ in `CPL` would be translated to $p \vee \neg p = 1$ in `IPL`, which is generally incoherent to an intuitionist. On the other hand, if we wanted to still preserve naturality, maybe we would opt to reject $\neg\neg(-) : \texttt{Heytalg} \to \texttt{Boolalg}$ as the best translation, accept $\Box(-)$, and explore what the topological semantics have to offer. But imposing the law of excluded middle on `IPL` is the simplest way to translate

IPL into CPL, which is reflected by the simplicity of the functor $\neg\neg(-)$ and its left adjoint U. So it does not seem plausible to reject $\neg\neg(-)$ on the basis of unnaturality.

From the categorical framework we have developed for $\neg\neg(-)$ and $\Box(-)$, we have the correct way to interpret the relationship between the translations. That is, as incompatible translations. The double-negation translation from IPL to CPL works because CPL can be embedded into IPL, just as how Boolean algebras are Heyting algebras. The modal translation from CPL to IPL does not work in the same way as the double-negation translation, because IPL requires some additional structure than just the CPL structure, just like how Boolean algebras are unequipped to become general Heyting algebras based on their algebraic semantics alone.

Overall, the categorical framework we developed to model CPL and IPL allowed us to explicitly see why they cannot be naturally translated between one another. We showed that $\neg\neg(-) : \texttt{Heytalg} \to \texttt{Boolalg}$ arises as the natural left adjoint to the forgetful functor $U : \texttt{Boolalg} \to \texttt{Heytalg}$, meaning $\neg\neg(-)$ arises naturally from the fact that Boolalg can be embedded into Heytalg. On the other hand, $\Box(-) : \texttt{Boolalg} \to \texttt{Heytalg}$ requires an additional assumption about the structure of Heytalg. In particular, it necessarily requires that we can define a topology on any lattice, which in turn requires $\Box(-)$ to not be a purely algebraic functor, but rather a functor on Top. Though, the fact that Heytalg requires an additional type of structure over Boolalg is interesting, and would not be explicitly evident without the category theory we developed.

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