Elaborating Cases of Ceresa Cycles Over Modular Curves

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Outline

1. What is Algebraic Geometry?
2. Cycles and Equivalences
3. Ceresa Cycles

Algebraic geometry is a branch of mathematics using algebraic tools to study geometric questions. In modern algebraic geometry, we study objects called *varieties*.

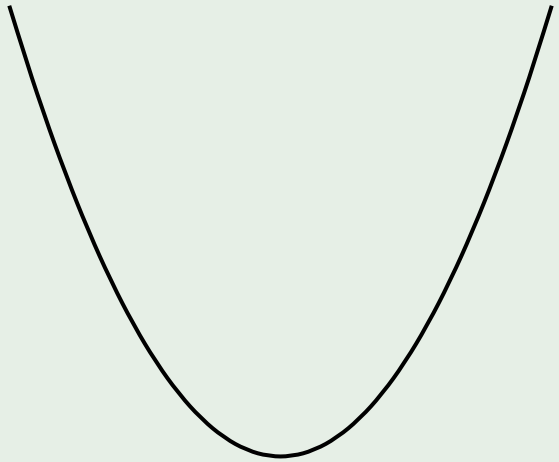
Definition

An *(algebraic) variety* is an (irreducible) set of solutions to a system of polynomials.

We will be considering one-dimensional varieties, which are called *curves*.

Example

Let $y = x^2$. Then the variety $V = \{(x, y) : y = x^2\}$ can be viewed as a parabola.



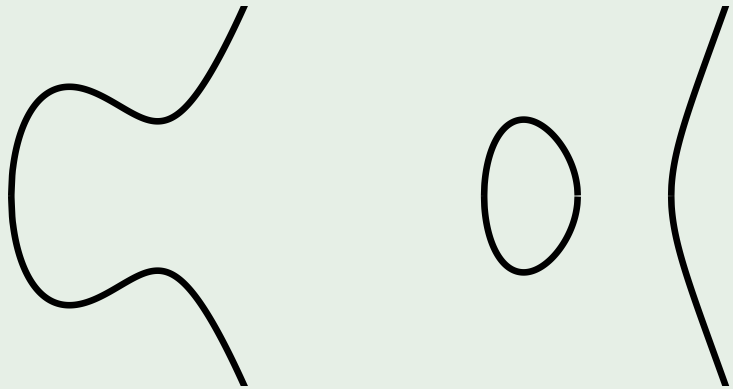
Example

Both

$$V_1 = \{(x, y) : y^2 = x^3 - 2x + 3\} \text{ (left)}$$

$$V_2 = \{(x, y) : y^2 = x^3 - 3x\} \text{ (right)}$$

are varieties.



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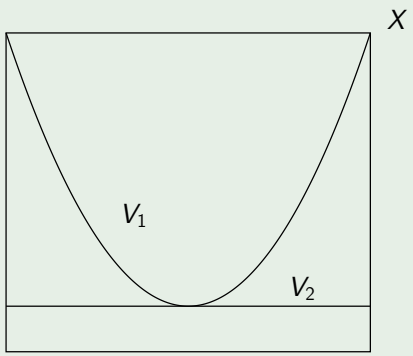
Let X be a variety. A 1-*cycle* on X is a finite formal sum

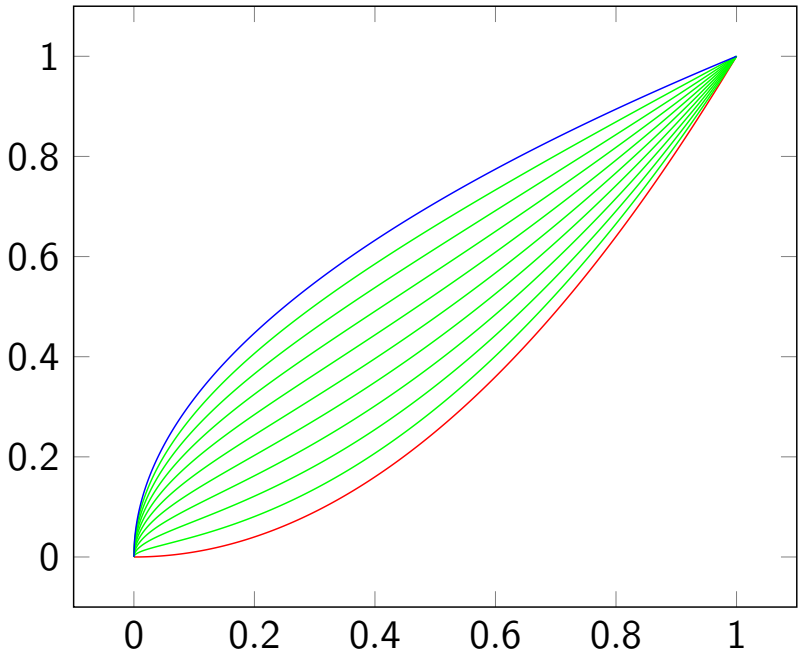
$$\sum_{i \in I} n_i [C_i],$$

where C_i are curves on X and $n_i \in \mathbb{Z}$.

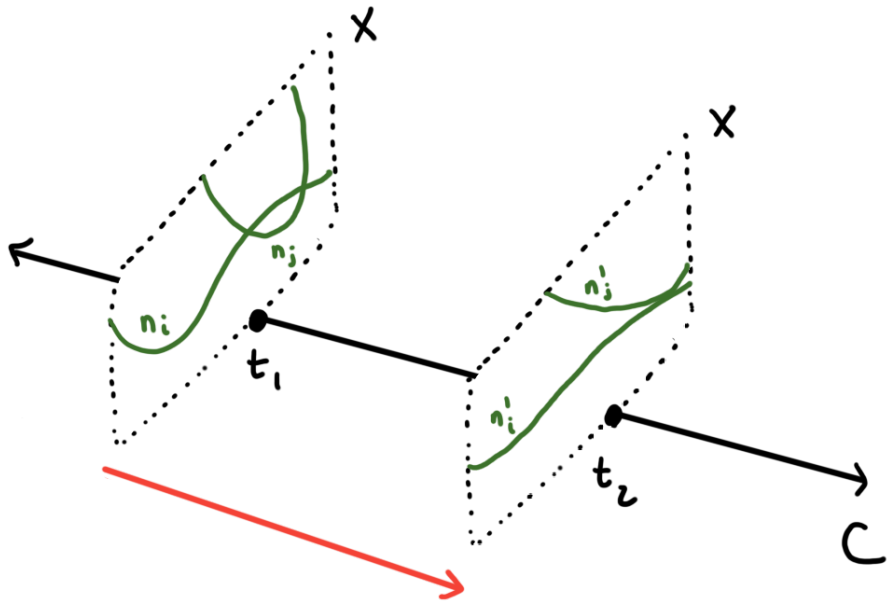
Example

For V_1, V_2, an algebraic cycle is $2[V_1] - 3[V_2]$.





- **Homological equivalence:** A cycle can be *continuously* deformed into another cycle.



- **Rational equivalence:** For $C = \mathbb{P}^1$, a cycle at $t_1 = 0$ can be *algebraically* deformed to a cycle at $t_2 = \infty$.
- **Algebraic equivalence:** For C any algebraic curve, a cycle at t_1 can be *algebraically* deformed into a cycle at t_2.

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Let C be a curve of genus g. We define the *Jacobian* of C to be

$$\mathrm{Jac}(C) = \frac{H^0(C, \Omega^1_{\mathrm{hol}}(C))^*}{H_1(C, \mathbb{Z})} \cong \frac{\mathbb{C}^g}{\Lambda},$$

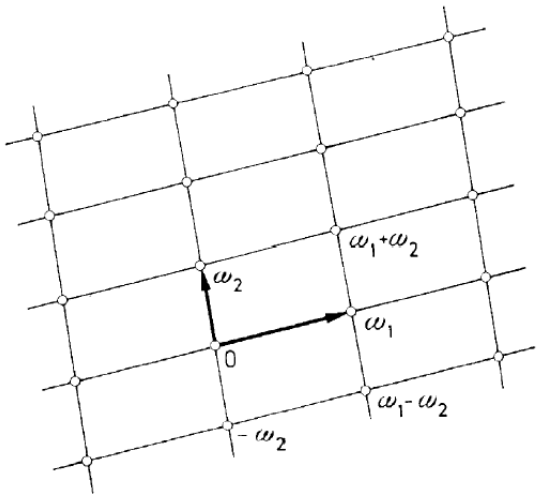
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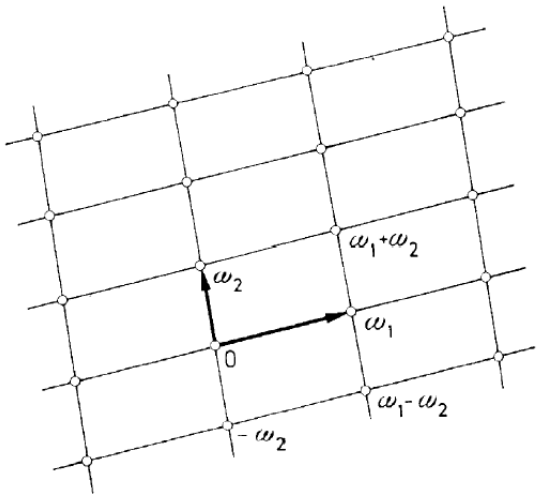


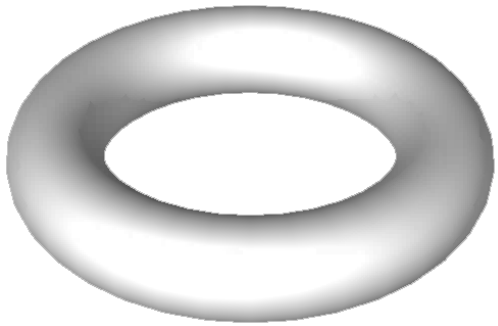
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Definition

Let C be a curve of genus g. Let $e \in C$ be fixed. We define the *Abel-Jacobi map* ι_e to be

$$\begin{aligned} \iota_e : C &\longrightarrow \mathrm{Jac}(C) \\ P &\longmapsto \left(\int_e^P \omega_1, \ldots, \int_e^P \omega_g \right) \mod \Lambda \end{aligned}$$

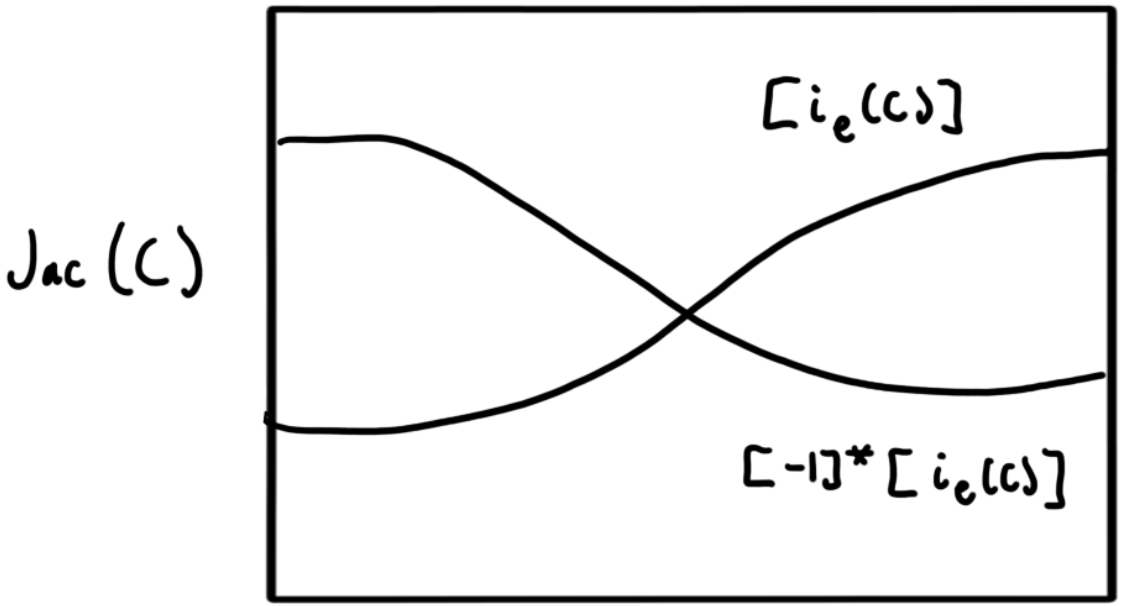
This map allows us to embed a curve inside it's own Jacobian! In particular, $\iota_e(C)$ is a curve inside $\mathrm{Jac}(C)$.

Definition

Let C be a smooth, projective, and geometrically integral curve. Let $e \in C$ be fixed. The *Ceresa cycle*[a] of C at e is the cycle

$$\mathrm{Cer}(C, e) = [\iota_e(C)] - [-1]^*[\iota_e(C)].$$

[a] Visualization courtesy of [Sri23, 12:54].



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When is Cer(C,e) $\neq$ 0 in $\mathrm{CH}_1(\mathrm{Jac}(C))$?

Thank you!

Ceresa Cycles of $X_0(N)$

One interesting family of curves are called *modular curves*, which have a very nice relationship with elliptic curves. One modular curve we like to study is $X_0(N)$, where $N > 0$.

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In number theory, we like to consider prime numbers, since we can express every integer uniquely as a product of primes. So, letting $\mathrm{Cer}(N)$ be the Ceresa cycle of $X_0(N)$, we ask:

For which primes p is Cer(p) $\neq$ 0 in $\mathrm{CH}_1(\mathrm{Jac}(X_0(p)))$?

Ceresa Cycles of $X_0(N)$

Theorem ([LR25])

If $p > 71$ or $p \in \{43, 53, 61, 67\}$, then $Cer(p) \neq 0$.

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There are finitely-many N such that $Cer(N) = 0$. In particular, if

$$N > 2^5 \times 3^4 \times 5^2 \times 7^2 \times \prod_{\substack{11 \leq p \leq 71 \\ p \notin \{43,53,61,67\}}} p,$$

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What about for non-primes less than the bound?

References

[Cer83] G. Ceresa. "C is not algebraically equivalent to C^- in its Jacobian". English. In: *Ann. Math. (2)* 117 (1983), pp. 285–291. ISSN: 0003-486X. DOI: 10.2307/2007078.

[LR25] Elvira Lupoian and James Rawson. *Ceresa Cycles of $X_0(N)$*. Preprint, arXiv:2501.14060 [math.AG] (2025). 2025. URL: https://arxiv.org/abs/2501.14060.

[Sri23] Padmavathi Srinivasan. *Computing Ceresa classes of curves*. 2023. URL: https://www.carmin.tv/en/video/computing-ceresa-classes-of-curves.