The Galois Correspondence of Covering Spaces

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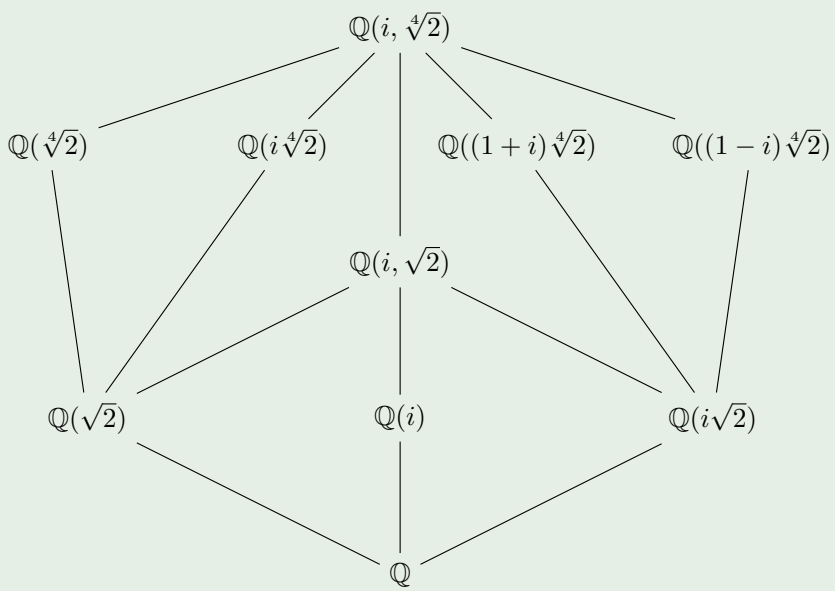
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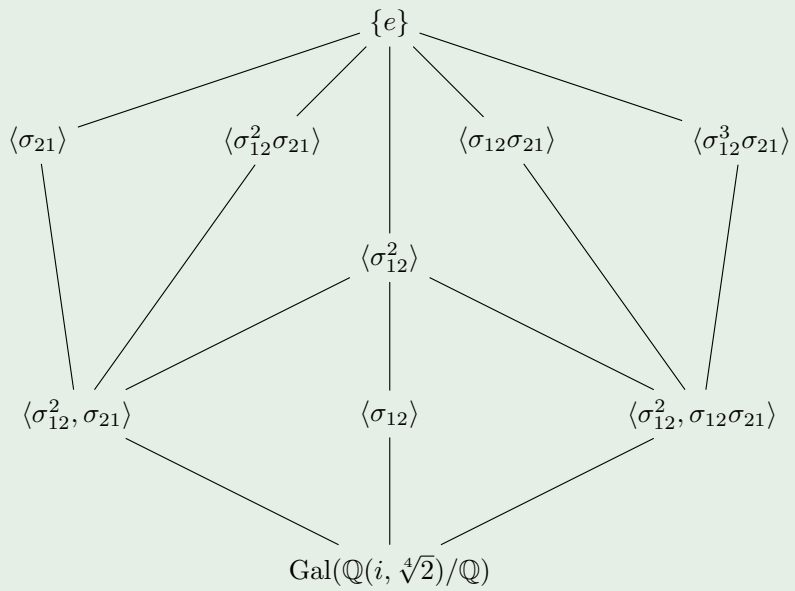
The Punchline of Galois Theory

Theorem ([Art98] The Fundamental Theorem of Galois Theory)

Let L/K be Galois and let $K \subseteq F \subseteq L$. Then there is a unique subgroup G of $Gal(L/K)$ such that $F = L^G$.

Example





Some Algebraic Topology

Let X be a connected space. We are interested in looking at loops at a fixed basepoint $x \in X$, or continuous paths that start and end at x. We denote the set of loops based at x by $\Omega(X, x, x)$.

Definition

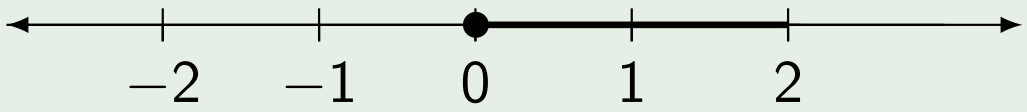
We say $\alpha, \beta \in \Omega(X, x, x)$ are *path homotopic* if there exists a continuous deformation from α to β. In this case we write $\alpha \sim \beta$.

Proposition

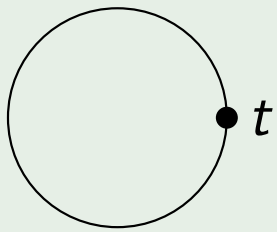
The set $\pi_1(X, x) = \Omega(X, x, x)/\sim$ is a group under path concatenation. We call this group the fundamental group.

Example

$$\pi_1(\mathbb{R}, 0) \cong \{\mathrm{id}_0\}$$



$$\pi_1(S^1, t) \cong \mathbb{Z}$$



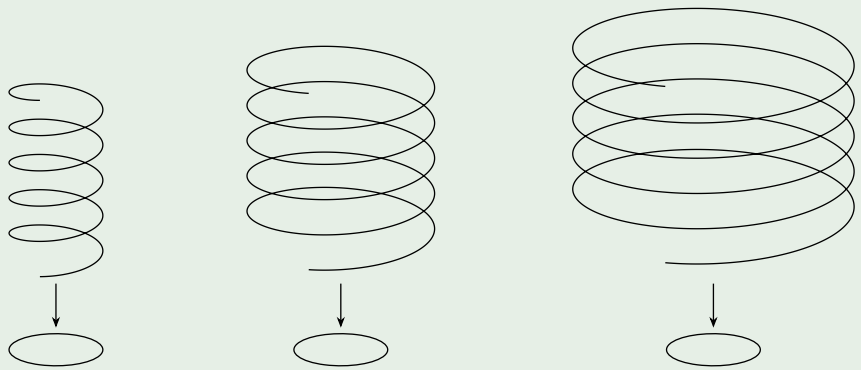
Covering Spaces

Definition

Let X be a connected space. A *cover* of X is a space E with a continuous map $p : E \to X$ such that for all $x \in X$, there is an open neighborhood $U \subset X$ such that $p^{-1}(U)$ is a disjoint union of open sets U_i such that $p : U_i \to U$ is a homeomorphism.

Example

A covering space of S^1 is $\mathbb{R}/n\mathbb{Z}$ with the map $t + n\mathbb{Z} \mapsto e^{2i\pi t}$. It can be visualized by "coiling" $\mathbb{R}$ with some period n:



Covering Spaces

Definition

For $p : E \to X$, if $\pi_1(E, e)$ is trivial, then we say connected E is *simply connected* and call E the *universal cover*.

The group of automorphisms of the universal cover E, denoted $\mathrm{Aut}(E)$, acts transitively on the fibers in E [Man15, pp. 232] and

$$\mathrm{Aut}(E) \cong \pi_1(X, x).$$

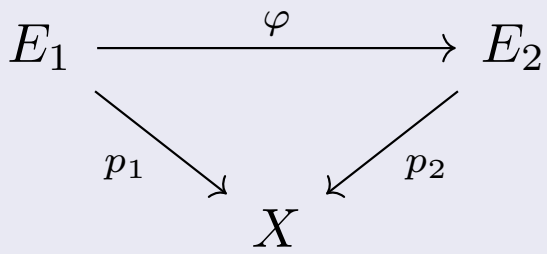
Example

$p : \mathbb{R} \to S^1$ is the universal cover of S^1.

Covering Spaces

Definition

Let X be path-connected. Denote $\mathbf{Cov}(X)$ to be the *category of covering spaces of* X, which has objects as (pointed) covers and morphisms as covering morphisms, i.e., φ such that the diagram commutes:



A covering automorphism is a covering morphism $E_1 \to E_1$.

Orbit Categories

Definition

For a group G, we denote $\mathcal{O}(G)$ to be the *category of orbits of* G, which has objects G-sets G/H and morphisms as G-maps $G/H \to G/K$ taking $gH \mapsto g\gamma K$, where $\gamma \in G$ such that $\gamma^{-1}H\gamma \subset K$.

Equivalently, we can view this category as the category of subgroups of G with morphisms as subconjugacy relations.

Galois Correspondence

Theorem (Galois Correspondence of Covering Spaces)

Let X be a connected, locally path connected, and semi-locally simply connected topological space. Fix $x \in X$. Then $\mathcal{O}(\pi_1(X, x))$ is equivalent to $\mathbf{Cov}(X)$.

These conditions on X imply that X has a universal cover [May99, pp. 30]. In fact, they are equivalent conditions [Man15, pp. 234-236].

Note: We will only show the key moves in this proof.

Galois Correspondence (Proof)

Proof.

Let $G = \pi_1(X, x)$. Let $p : E \to X$ be the universal cover. **We will make the identification $\mathbf{Aut}(E) = G$ for the rest of this proof and take $g \in G$ to be an automorphism of E.**
Define the following functor:

$$\begin{aligned} \mathcal{E}(-) : \mathcal{O}(G) &\longrightarrow \mathbf{Cov}(X) \\ G/H &\longmapsto E/H \\ (gH \mapsto g\gamma K) &\longmapsto (He \mapsto K\gamma e) \end{aligned}$$

Action on morphisms are defined since $\gamma \in G$ such that $\gamma^{-1} H \gamma \subset K$, so we need to check that this functor is well-defined on objects.

Galois Correspondence (Proof)

Proof (cont).

Consider $\mathcal{E}(G/H) = E/H$. Take the quotient map $q : E \to E/H$. If $U \ni x$ is a fundamental neighborhood, then

$$p^{-1}(U) = \bigsqcup_{g \in G} gV$$

for connected $V \subset p^{-1}(U)$. Under $q(p^{-1}(U))$, each hV for $h \in H$ are identified, so

$$q(p^{-1}(U)) = \bigsqcup_{g \in G/H} g(q(V)).$$

Hence, $E/H \to X$ is a cover and $\mathcal{E}(-)$ is well-defined. $\square$

Galois Correspondence

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Corollary

Let X be a connected, locally path connected, and semi-locally simply connected topological space. Fix $x \in X$. Then for every subgroup H of $\pi_1(X, x)$, there is a cover $p : E \to X$ with a basepoint $\tilde{x} \in p^{-1}(x)$ such that

$$p_*(\pi_1(E, \tilde{x})) = H.$$

References

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[Man15] Marco Manetti. *Topology. Translated from the Italian by Simon G. Chiossi*. Vol. 91. Unitext. Cham: Springer, 2015. ISBN: 978-3-319-16957-6; 978-3-319-16958-3.

[May99] J. P. May. *A concise course in algebraic topology*. Chicago, IL: University of Chicago Press, 1999. ISBN: 0-226-51182-0; 0-226-51183-9. URL: https://www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf.