Group Schemes

Leonardo Finzi

University of Pittsburgh
Department of Mathematics & Department of Philosophy

Advisor: Eugene Eyeson

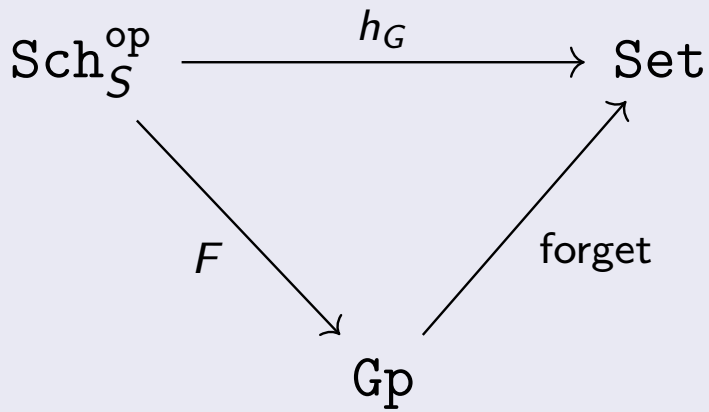
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Group Schemes

Definition

Let S be a scheme. A **group scheme** over S (or an S-**group**) is a functor $F : \mathtt{Sch}_S^{\mathrm{op}} \to \mathtt{Gp}$ and an S-scheme G such that the following diagram commutes:



Note that h_G denotes the Yoneda functor which takes

$$X \overset{h_G}{\longmapsto} \mathtt{Sch}_S^{\mathrm{op}}(X, G).$$

Affine Group Schemes

Example (Additive Group Scheme over k)

For $S = \operatorname{Spec} k$, $\mathbb{A}^n_S$ is an S-group. If $R \in k\text{-}\mathtt{alg}$, then

$$\mathbb{A}^n_S(R) = R^n.$$

Additionally, $\mathbb{A}^n_S$ is represented by $\operatorname{Spec} k[x_1, \ldots, x_n]$.

Example

For a field k, GL_n is a k-group scheme such that for all $R \in k\text{-}\mathtt{alg}$,

$$\mathrm{GL}_n(R) = \{A \in \mathcal{M}_n(R) : \det A \notin R^*\}.$$

Additionally, GL_n is represented by

$$\operatorname{Spec} k\left[x_{ij}, \frac{1}{\det x_{ij}} : 1 \leq i, j \leq n\right].$$

Affine Group Schemes

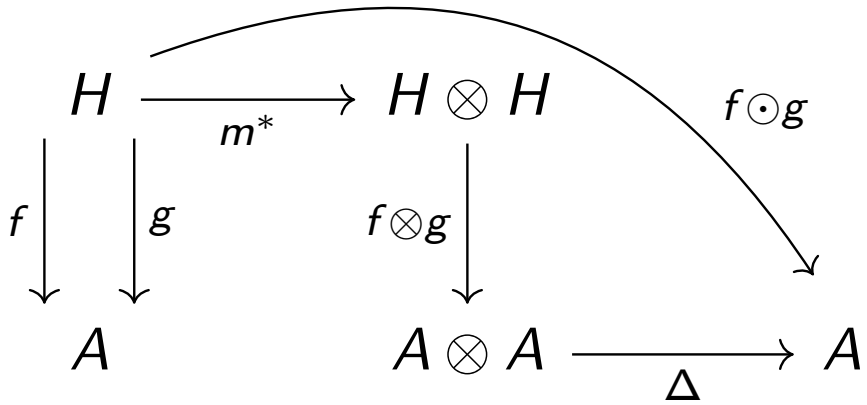
Schemes are nice because there is a categorical anti-equivalence between the category of affine schemes and the category of commutative rings. As it turns out, affine S-groups can be nicely characterized as well.

Theorem

There is a categorical anti-equivalence between the **category of affine S-groups** *and the* **category of commutative quasi-coherent Hopf $\mathcal{O}_S$-algebras**.

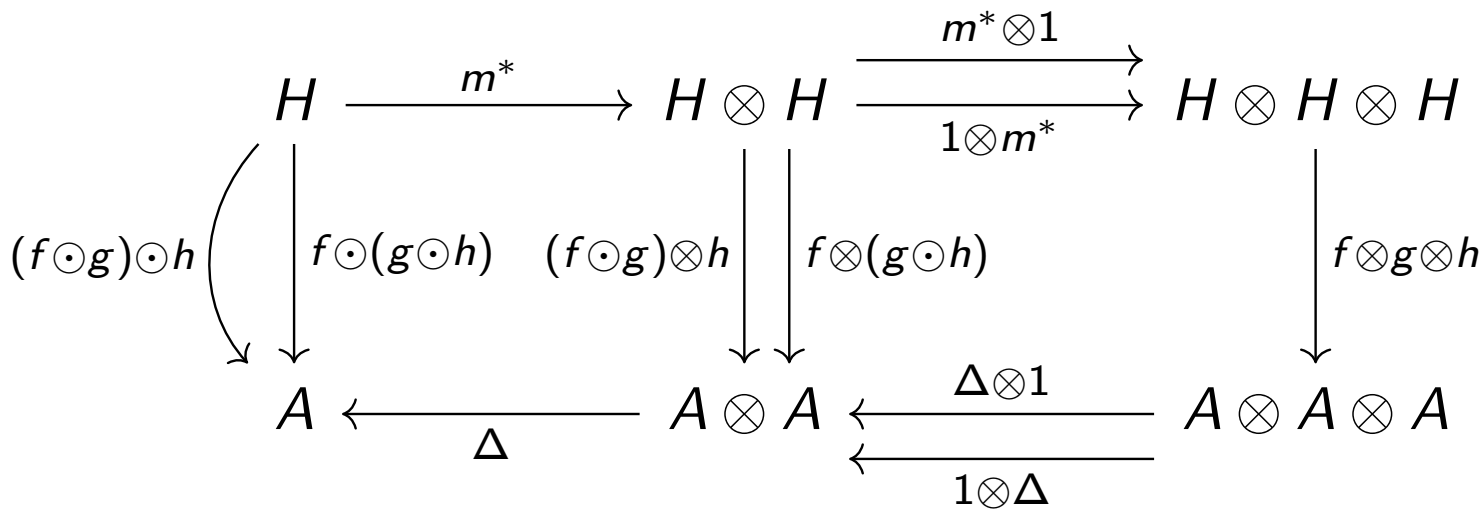
Hopf Algebras to Affine Group Schemes

Suppose $S = \text{Spec } R$. Let (H, m^*, e^*, ι^*) be a Hopf R-algebra. Let $G = \text{Spec } H$ and $A \in R\text{-}\mathtt{alg}$. We define $f \odot g$ below:



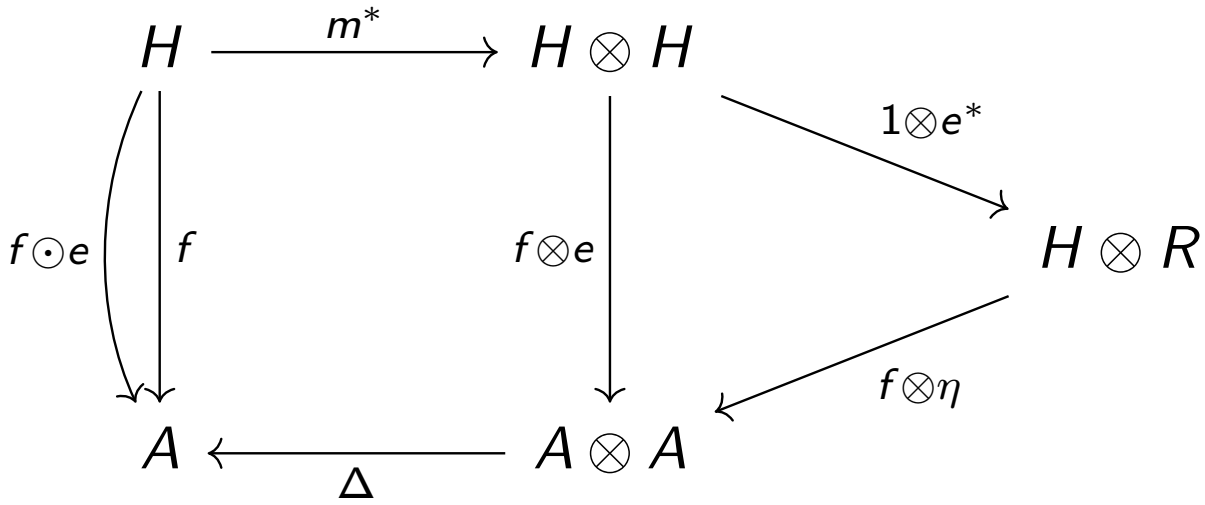
Hopf Algebras to Affine Group Schemes

We get associativity:



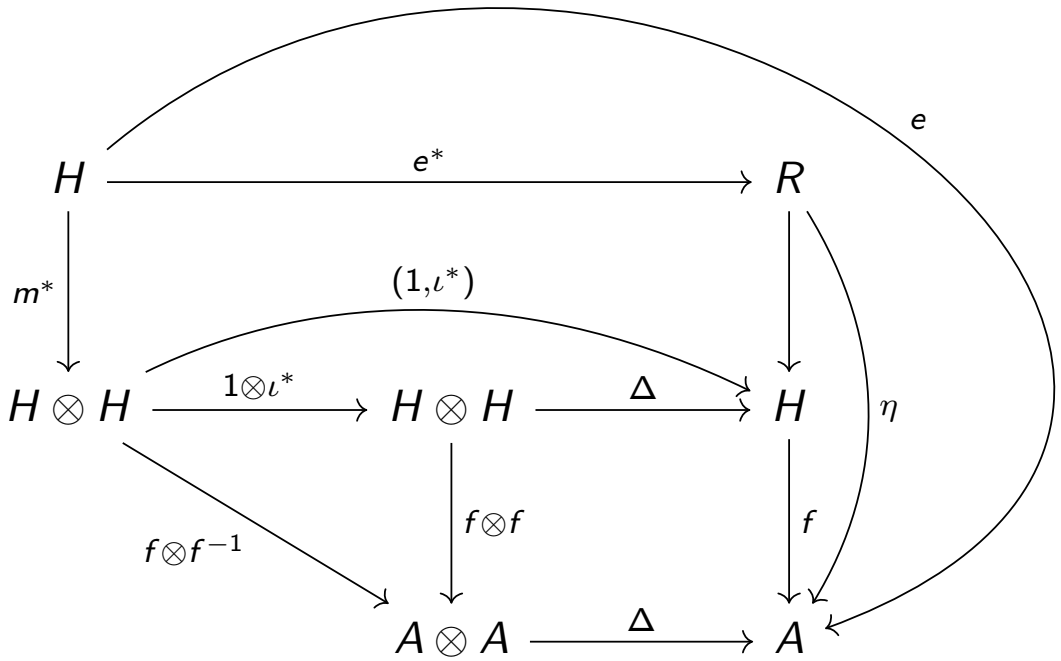
Hopf Algebras to Affine Group Schemes

Let $e = \eta \circ e^*$ with $\eta : R \to A$ the structure map. Then e is the identity:



Hopf Algebras to Affine Group Schemes

Finally, we get that $f^{-1} = f \circ \iota^*$ is the inverse of f:



Abelian k-Varieties

Definition

For k a field, a k-**abelian variety** is a connected, proper, and smooth commutative group scheme over k.

- Elliptic curves (smooth curves of degree 3 in $\mathbb{P}^2$) are examples of abelian k-varieties.

Definition

Let $A \in \mathtt{AV}_k$. We define the **dual abelian k-variety** to be

$$A^t := \mathrm{Pic}^0 A,$$

for $\mathrm{Pic}^0 A = \{\mathcal{L} \in \mathrm{Pic}\ A : t_a^* \mathcal{L} \cong \mathcal{L}\}$ where $t_a : A \xrightarrow{\sim} A$ takes $x \mapsto x + a$.

An Extreme Theorem

Theorem

Let $A \in \mathcal{A}V_k$. If $\mathcal{L} \ncong \mathcal{O}$ in $Pic^0 A$, then $H^i(A, \mathcal{L}) = 0$ for all $i \geq 0$.

Proof.

Let $\mathcal{L}$ be non-trivial. Suppose $H^0(A, \mathcal{L}) = 0$ but suppose there is smallest n such that $H^n(A, \mathcal{L}) \neq 0$. For the multiplication map $m : A \times A \to A$, $m^*\mathcal{L} = \mathcal{L} \boxtimes \mathcal{L}$ and $m^* : H^n(A, \mathcal{L}) \hookrightarrow H^n(A \times A, \mathcal{L} \boxtimes \mathcal{L})$ since $m \circ (\mathrm{id}, e) = \mathrm{id}$ implies $(\mathrm{id}, e)^* \circ m^* = \mathrm{id}$. The Künneth isomorphism gives

$$H^n(A \times A, \mathcal{L} \boxtimes \mathcal{L}) \cong \bigoplus_{i+j=n} (H^i(A, \mathcal{L}) \otimes H^j(A, \mathcal{L})).$$

The RHS is zero since $H^0 = H^1 = \cdots = H^{n-1} = 0$. Then m^* is the zero map and $\ker m^* = H^n$, which contradicts m^* injective. $\square$

Thank you!