FINAL EXAM : Math 1350 : Fall 2014

GOOD LUCK!!!

Problem 1. (5pts)
It is known that S is a smooth surface in $\mathbb{R}^3$ and that $f_1 : U_1 \to \mathbb{R}^3$ and $f_2 : U_2 \to \mathbb{R}^3$ are two local charts of S around some point $p \in S$, such that $f_1(0) = f_2(0) = p$. An absent-minded student from a differential geometry class computed the transition map $(f_2)^{-1} \circ f_1$ and scribbled the answer on a scrap of paper. Later, he found four pieces of paper with the following formulae:

(a) $u = x^2 + y + z, \quad v = e^{x - \sin y} z,$
(b) $u = \sqrt{x}, \quad v = x^{1/7},$
(c) $u = 20, \quad v = 2,$
(d) $u = -x + \frac{y}{x+1}, \quad v = \frac{e^y}{x+1},$
(e) $u = x + y, \quad v = 2x$

He knows that one of these five is the right answer and the other four are notes from a computational economics tutorial. Which of the formulae (a)–(e) represents the transition map? Explain your answers.

Problem 2. (15pts)
Calculate the Frenet frame, the curvature and torsion of the following curve in $\mathbb{R}^3$:

$$\gamma(t) = (\cos t, \sin t, t) \qquad t \in (-1, 1)$$

Problem 3. (20pts)
Let S be a surface of revolution with parametrization: $\sigma(u, v) = (\phi(u) \cos v, \phi(u) \sin v, u)$.
(1) Find the principal directions, the principal curvatures, and the Gaussian curvature of S.
(2) Show that any meridian curve $v \equiv v_0$ is a geodesic. Show that a horisontal curve $u \equiv u_0$ is a geodesic if and only if $\phi'(u_0) = 0$.

Problem 4. (20pts)
Write the formula for or disprove existence of a diffeomorphism between an open subset of the plane $\mathbb{R}^2$ and an open subset of the unit sphere $\mathbb{S}^2$ which:
(1) preserves the angles (conformal diffeomorphim).
(2) preserves the area (equiareal diffeomorphism).
(3) preserves the lengths of curves (isometry).

Problem 5. (20pts)
Let S be a compact surface in $\mathbb{R}^3$.
(1) Show that the Gauss map $\mathcal{G} : S \to \mathbb{S}^2$ is surjective.
(2) Show that $\int_S \max\{\kappa, 0\}\, \mathrm{d}\mathcal{A} \geq 4\pi$ (where κ denotes the Gaussian curvature of S).
(3) Deduce that for S with non-zero genus, there also must hold: $\int_S \max\{-\kappa, 0\}\, \mathrm{d}\mathcal{A} \geq 4\pi$.