Solutions to Midterm 2 — Marta Lewicka

Problem 1 (1) $\sigma_u = (-v\sin u, v\cos u, 1)$, $\sigma_v = (\cos u, \sin u, 0)$

$$\sigma_u \times \sigma_v = \begin{vmatrix} i & j & k \\ -v\sin u & v\cos u & 1 \\ \cos u & \sin u & 0 \end{vmatrix} = (-\sin u, \cos u, -v) \neq 0$$ hence regular parametris.

$$\vec{N} = \frac{1}{\sqrt{1+v^2}}(-\sin u, \cos u, -v)$$

ruled surface: $\sigma(u,v) = \underbrace{(0,0,u)}_{\text{leading curve}} + v\underbrace{(\cos u, \sin u, 0)}_{\text{direction of the rule}}$.

(2) $\sigma_{uu} = (-v\cos u, -v\sin u, 0)$
$\sigma_{uv} = (-\sin u, \cos u, 0)$
$\sigma_{vv} = (0,0,0)$

$$I_S = \begin{bmatrix} v^2+1 & 0 \\ 0 & 1 \end{bmatrix}, \quad II_S = \begin{bmatrix} 0 & \frac{1}{\sqrt{1+v^2}} \\ \frac{1}{\sqrt{1+v^2}} & 0 \end{bmatrix}$$

$$I_S^{-1} II_S = \begin{bmatrix} \frac{1}{v^2+1} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & \frac{1}{\sqrt{1+v^2}} \\ \frac{1}{\sqrt{1+v^2}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{(v^2+1)^{3/2}} \\ \frac{1}{(v^2+1)^{1/2}} & 0 \end{bmatrix}$$

Hence: $K = \det(I_S^{-1} II_S) = -\frac{1}{(v^2+1)^2}$, $H = \frac{1}{2}\mathrm{tr}(I_S^{-1} II_S) = 0$.

Problem 2 (1) $\gamma : (-\alpha, \alpha) \to S$ such that $\gamma''(t) \parallel \vec{N}(\gamma(t))$ $\forall t$

constant speed: $\frac{d}{dt}|\gamma'(t)|^2 = 2\langle \gamma'(t), \gamma''(t)\rangle = 0$ since $\gamma' \perp \vec{N}$.

(2) $K_g = \langle \gamma'', \vec{N} \times \gamma' \rangle$. Since $\gamma'' \parallel \vec{N}$, and $\vec{N} \perp \vec{N} \times \gamma'$ it follows that $K_g = 0$

(3) $\gamma(t) = \left(3\cos\frac{t}{\sqrt{10}}, 3\sin\frac{t}{\sqrt{10}}, \frac{t}{\sqrt{10}}\right)$ so: $\gamma''(t) = \left(-\frac{3}{10}\cos\frac{t}{\sqrt{10}}, -\frac{3}{10}\sin\frac{t}{\sqrt{10}}, 0\right)$

$$\vec{N}(\gamma(t)) = \frac{1}{\sqrt{10}}\left(-\sin\frac{t}{\sqrt{10}}, \cos\frac{t}{\sqrt{10}}, -3\right)$$

Since $\gamma'' \nparallel \vec{N}(\gamma(t))$, γ is not a geodesic on S.

Problem 3 Let $\sigma : U \to S$ be a parametrisation of S.

Then $\sigma^\lambda(u,v) = \sigma(u,v) + \lambda \vec{N}(u,v)$ is a parametrisation of S^λ.

(1) $\sigma^\lambda_u = \sigma_u + \lambda \vec{N}_u = \sigma_u - \lambda \mathcal{W} \sigma_u \in T_pS$

$\sigma^\lambda_v = \sigma_v + \lambda \vec{N}_v = \sigma_v - \lambda \mathcal{W} \sigma_v \in T_pS$

so $\sigma^\lambda_u, \sigma^\lambda_v$ are linearly independent for $|\lambda|$ small.

Hence $N^\lambda = N$, so S^λ is oriented.

(2) $\underbrace{\mathcal{W}^\lambda \sigma^\lambda_u}_{=\mathcal{W}^\lambda(Id - \lambda\mathcal{W})\sigma_u} = -\partial_u N^\lambda = -\partial_u N = \mathcal{W}\sigma_u$, $\underbrace{\mathcal{W}^\lambda \sigma^\lambda_v}_{=\mathcal{W}^\lambda(Id - \lambda\mathcal{W})\sigma_v} = -\partial_v N^\lambda = -\partial_v N = \mathcal{W}\sigma_v$

Hence: $\mathcal{W}^\lambda (Id - \lambda \mathcal{W}) = \mathcal{W} \iff \mathcal{W}^\lambda = \mathcal{W}(Id - \lambda\mathcal{W})^{-1}$.

(3) The principal curvatures of S^λ, denoted by k^λ, satisfy:

$$0 = \det(\mathcal{W}^\lambda - k^\lambda Id) = \det(\mathcal{W}(Id - \lambda\mathcal{W})^{-1} - k^\lambda Id) =$$

$$= \det(\mathcal{W} - k^\lambda(Id - \lambda \mathcal{W}))\ \det(Id - \lambda\mathcal{W})^{-1}$$

$$\iff 0 = \det(\mathcal{W} - k^\lambda(Id - \lambda\mathcal{W})) = \det((1 + \lambda k^\lambda)\mathcal{W} - k^\lambda Id)$$

$$\iff 0 = \det\left(\mathcal{W} - \frac{k^\lambda}{1+\lambda k^\lambda} Id\right) = 0 \iff k = \frac{k^\lambda}{1+\lambda k^\lambda}$$

$\hookrightarrow$ principal curvatures of S

$$\iff k^\lambda = k + \lambda k k^\lambda$$

$$\iff k^\lambda = \frac{k}{1 - \lambda k}.$$

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