MIDTERM 1 : Math 1700 : Spring 2014
SOLUTIONS

Problem 1. (5+5 points)
Let (X,d) be a metric space.
(i) Show that if a ball in X of radius 7 is a subset of a ball of radius 3, then these balls must be the same.
(ii) Can a ball in X of radius 4, be a proper subset of a ball of radius 3?

(i) Let $B(x,7) \subset B(y,3)$. Take any $z \in B(y,3)$ and note that: $d(z,x) \leq d(z,y) + d(y,x) < 3+3 = 6 < 7$. Consequently: $B(y,3) \subset B(x,7)$.

(ii) Yes, this is possible. Let $X = \{-5/2, 0, 5/2\}$ be the subspace of $\mathbb{R}$, inheriting its (Euclidean) metric. Then $B(0,3) = X$ and $B(5/2,4) = \{0, 5/2\}$.

Problem 2. (10+5 points)
Let $\{X_\alpha\}$ be a family of topological spaces. For each α, let A_α be a subset of X_α.
(i) Prove that: $\overline{\prod_\alpha A_\alpha} = \prod_\alpha \bar{A}_\alpha$, where the closure in the left hand side is taken with respect to the product topology.
(ii) Is the same true for the box topology?

We will prove that the desired equality is valid in both box and product topologies. We first prove the inclusion "$\subset$". Take any $f \in \overline{\prod A_\alpha}$. Fix an index α_0; we want to show that $f(\alpha_0) \in \overline{A_{\alpha_0}}$. Take any open set $U_{\alpha_0} \subset X_{\alpha_0}$, so that $f(\alpha_0) \in U_{\alpha_0}$. Consider the set $U = \prod U_\alpha$, where U_{α_0} is given and $\forall \alpha \neq \alpha_0$ we put $U_\alpha = X_\alpha$. Then U is open in both topologies and it contains f. Therefore there exists $g \in \prod A_\alpha \cap \prod U_\alpha$, and consequently $g(\alpha_0) \in A_{\alpha_0} \cap U_{\alpha_0}$. Hence $f(\alpha_0) \in \overline{A_{\alpha_0}}$.

We now prove the inclusion "$\supset$". Given $f \in \prod \overline{A_\alpha}$, take any basis element $\prod U_\alpha$ in the topology under consideration, which contains f. Then for all indices α, we have: $f(\alpha) \in \overline{A_\alpha} \cap U_\alpha$. Hence there exists $g(\alpha) \in A_\alpha \cap U_\alpha$. Consequently: $g \in \prod A_\alpha \cap \prod U_\alpha$, which implies that $f \in \overline{\prod A_\alpha}$.

Problem 3. (5+10 points)
Let X, Y be two topological spaces, and let $f : X \to Y$ be a function.
(i) Prove that if f is continuous, then for every convergent sequence $x_n \to x_0$ in X, the sequence $f(x_n)$ converges to $f(x_0)$ in Y.
(ii) Is the converse true?

(i) Let U be an open set in Y containing $f(x_0)$. Since $f^{-1}(U)$ is open, there exists N such that $x_n \in f^{-1}(U)$ for all $n \geq N$. Therefore $f(x_n) \in U$ for all $n \geq N$, which establishes the claim.

(ii) The converse is not true. Consider the topological space $X = \mathbb{R}$ with the topology given by the empty set, and the sets whose complements are countable. Let $Y = \mathbb{R}$ be standard Euclidaen topological space, and let $f : X \to Y$ be the identity map. Notice first that f is not continuous, because $f^{-1}((0,\infty)) = (0,\infty)$ is not open in X.

On the other hand, f is sequentially continuous. Indeed, a sequence x_n converges to x_0 in X if and only if there exists N such that $x_n = x_0$ for all $n \geq N$. Clearly, $f(x_n) = x_n$ converges then to $f(x_0) = x_0$ in Y.

Problem 4. (10 points)
Let X and Y be two topological spaces and let Y be Hausdorff. Given a function $f : X \to Y$, define G_f (called the graph of f) to be the subspace:

$$G_f = \{(x, f(x));\ x \in X\}$$

of $X \times Y$. Prove that if f is continuous, then G_f is closed.

We will show that $(X \times Y) \setminus G_f$ *is open. Take* $(x_0, y_0) \notin G_f$*, so that* $y_0 \neq f(x_0)$*. Since* Y *is Hausdorff, there exists two open disjoint sets* U_1 *and* U_2*, such that* $y_0 \in U_1$ *and* $f(x_0) \in U_2$*. Then the set* $f^{-1}(U_2) \times U_1$ *is open in* $X \times Y$ *(in view of continuity of* f*) and it contains the point* (x_0, y_0)*. Note also that for every* $x \in f^{-1}(U_2)$ *there is:* $f(x) \in U_2$ *so* $f(x) \notin U_1$*. Consequently:* $f^{-1}(U_2) \times U_1 \subset (X \times Y) \setminus G_f$*.*

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Bonus Problem. (15 points)
Prove that the cartesian product of an arbitrary family of connected topological spaces is connected in the product topology.

Let $\{X_\alpha\}_{\alpha \in A}$ *be a family of connected topological spaces. We will show that the space* $X = \prod_{\alpha \in A} X_\alpha$ *is connected.*

For every $\alpha \in A$*, choose a point* $a_\alpha \in X_\alpha$*. Given any finite subset* $B \subset A$*, consider the following subspace of* X*, which is homeomorpic to* $\prod_{\alpha \in B} X_\alpha$*, hence connected (we use here the theorem that the Cartesian product of finitely many connected spaces is connected):*

$$Z_B = \prod_{\alpha \in A} C_\alpha, \quad \textit{where} \;\; C_\alpha = \begin{cases} X_\alpha & \textit{if } \alpha \in B \\ \{a_\alpha\} & \textit{if } \alpha \notin B \end{cases}.$$

Note that:

$$\bigcap_{B \subset A;\ B \textit{ finite}} Z_B = \prod_{\alpha \in A} \{a_\alpha\}.$$

When the intersection of the connected spaces is nonempty, their union is connected, so the space:

$$Z = \bigcup_{B \subset A;\ B \textit{ finite}} Z_B$$

is a connected subspace of X*.*

Now we will show that $\bar{Z} = X$*, which will end the proof of connectivity of* X*. Take any point* $\prod\{x_\alpha\} \in X$ *and take its arbitrary open neighbourhood* V *in the produt topology in* X*. The set* V *must contain another open neighbourhood* U *of the same point, of the form:* $U = \prod_{\alpha \in A} U_\alpha$*, where for all* α *outside of a finite set* $B \subset A$ *we have* $U_\alpha = X_\alpha$*. Then:*

$$(Z_B \cap U) \ni \prod_{\alpha \in A} \{c_\alpha\}, \quad \textit{where} \;\; c_\alpha = \begin{cases} x_\alpha & \textit{if } \alpha \in B \\ a_\alpha & \textit{if } \alpha \notin B \end{cases}.$$

Hence: $Z \cap U \neq \emptyset$*, and it follows that:* $\prod\{x_\alpha\} \in \bar{Z}$*. This ends the proof.*