MIDTERM 2 : Math 1700 : Spring 2014
SOLUTIONS

Problem 1. (10 points)
Let X be a compact topological space and for each $i \in \mathbb{N}$ let $F_i \subset X$ be a closed, nonempty subset of X. Assume further that the subsets F_i are nested, i.e.:

$$\forall i \quad F_{i+1} \subset F_i.$$

Using only the definition of compactness, prove that the set $\bigcap_{i=1}^{\infty} F_i$ is nonempty.

Assume, by contradiction, that $\bigcap_{i=1}^{\infty} F_i = \emptyset$. *Then, the sequence of open sets* $U_n = X \setminus F_n$ *is a covering of* X. *Notice that this sequence is increasing, i.e:* $U_n \subset U_{n+1}$. *By compactness of* X, $\{U_n\}$ *has a finite subcovering:* $X = \cup_{i=1}^{N} U_n$. *But this implies that* $U_N = X$, *contradiction with* $F_N \neq \emptyset$.

Problem 2. (10 points)
Let X and Y be two topological spaces. Show that $X \times Y$ (with the product topology) is separable if and only if X and Y are both separable.

"⇒": Let $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ *be a dense countable subset of* $X \times Y$. *Then* $\{a_n\}_{n\in\mathbb{N}}$ *is a dense countable subset of* X. *To prove the density, take an open set* $U \subset X$.*Then* $U \times Y$ *is open in* $X \times Y$, *so it contains some* (a_n, b_n). *Consequently,* $a_n \in U$. *Separability of* Y *is proven in the same manner.*

"⇐": Let $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ *be dense countable subsets of, respectively,* X *and* Y. *Then* $\{(a_m, b_n)\}_{m,n\in\mathbb{N}}$ *is a dense countable subset of* $X \times Y$. *To prove the density, let* $U \times V$ *be any basis open subset of* $X \times Y$. *Then for some* m, n *we have:* $a_m \in U$ *and* $b_n \in V$. *Consequently,* $(a_m, b_n) \in U \times V$.

Problem 3. (10 points)
Let X be a topological space which is completely regular (i.e. $T_{3\frac{1}{2}}$). Let $A, B \subset X$ be two closed, disjoint subsets, and assume that A is compact. Prove that there exists a continuous function $f : X \to [0,1]$ such that $f_{|A} = 0$ and $f_{|B} = 1$.

Since X *is* $T_{3\frac{1}{2}}$, *for every* $a \in A$ *there existsa continuous function* $f_a : X \to [0,1]$ *such that* $f_a(a) = 0$ *and* $(f_a)_{|B} = 1$. *Consider the open sets* $U_a = (f_a)^{-1}(-\infty, \frac{1}{2})$, *which cover the set* A. *By compactness, there exists a finite subcover* $\{U_{a_i}\}_{i=1}^{N}$.

Define the function $g : X \to [0,1]$ *by:* $g(x) = \prod_{i=1}^{N} f_{a_i}(x)$. *Clearly,* g *is continuous and* $g_{|B} = 1$. *Further, for every* $x \in A$ *there is* $i : 1..N$ *so that* $f_{a_i}(x) < \frac{1}{2}$, *so that:* $g(x) < \frac{1}{2}$. *We hence see that* $g(A) \subset [0, \frac{1}{2}]$.

Let $h : [0,1] \to [0,1]$ *be a continuous function such that* $h_{|[0,1/2]} = 0$ *and* $h(1) = 1$. *Then* $f := h \circ g : X \to [0,1]$ *is a continuous function with the desired properties.*

Problem 4. (10 points)
Show that if X is a locally compact Hausdorff space then its one-point compactification is also Hausdorff.

We know that any one-point compactification of X is homeomorphic to the space $Y = X \cup \{a\}$ where the topology is given by: the collection of U open subsets of X and sets $\{a\} \cup (X \setminus C)$ for all compact subsets C of X.

It is enough to show that Y is Hausdorff. Take $x \neq y$ in Y. We have to show that x, y can be separated by two disjoint open sets in Y. In the first case: $x, y \in X$. Using X being Hausdorff, there are two disjoint open sets in X (hence open sets in Y) such that $x \in U$ and $y \in V$.

In the second case $x = a$ and $y \in X$. By local compactness of X, there is an open set U and a compact set C such that $y \in U \subset C \subset X$. Then $Y \setminus C$ is an open neighbourhood of a in Y, which is disjoint from U. This ends the proof.

Problem 5. (10 points)
Let (X, d) be a metric space.
(i) If X is Lindelöf, show that X is second countable.
(ii) If X is separable, show that X is second countable.

(i) For every $n \in \mathbb{N}$ consider the following open cover of X: $\{B(x, \frac{1}{n})\}_{x \in X}$. By the Lindelöf property, choose a countable subcover $\mathcal{B}_n$. Then $\mathcal{B} := \cup_{n=1}^{\infty} \mathcal{B}_n$ is countable and it generates a (countable) basis of the metric topology in X. To prove this last property, fix $x \in U$ an open subset of X. Clearly $B(x, \epsilon) \subset U$, for some $\epsilon > 0$. By construction of $\mathcal{B}$, there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that $x \in B(x_n, \frac{1}{n})$ for all $n \in \mathbb{N}$. Consequently, $d(x_n, x) < \frac{1}{n}$ and so $x_n \to x$ as $n \to \infty$. In particular, there must be: $B(x_n, \frac{1}{n}) \subset B(x, \epsilon) \subset U$ for $n > \frac{2}{\epsilon}$ large enough, because:

$$\forall y \in B(x_n, \frac{1}{n}) \qquad d(y, x) \leq d(y, x_n) + d(x_n, x) < \frac{1}{n} + \frac{1}{n} < \epsilon.$$

(ii) Let $\{x_n\}_{n \in \mathbb{N}}$ be a countable dense subset of X. Then $\{B(x_n, \frac{1}{m});\ n, m \in \mathbb{Z}\}$ is countable and it generates a (countable) basis of the metric topology in X. To prove this last property, fix $x \in U$ an open subset of X. Clearly $B(x, \epsilon) \subset U$, for some $\epsilon > 0$. Take n, m such that $d(x, x_n) < \frac{\epsilon}{2}$ and $m > \frac{2}{\epsilon}$. Then $B(x_n, \frac{1}{m}) \subset B(x, \epsilon) \subset U$, because:

$$\forall y \in B(x_n, \frac{1}{m}) \qquad d(y, x) \leq d(y, x_n) + d(x_n, x) < \frac{1}{m} + d(x_n, x) < \epsilon.$$

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Bonus Problem. (20 points) (Only complete/almost-complete solution to this problem will be awarded points)
Let (X, d) be a metric space. Show that X is compact if and only if every continuous function $f : X \to \mathbb{R}$ is bounded.

"$\Rightarrow$": Let $f : X \to \mathbb{R}$ be a continuous function. Since X is compact, $f(X)$ is compact. Since compact subsets of $\mathbb{R}$ are bounded, f is bounded.

"$\Leftarrow$": Suppose that the metric space X is not compact, so that there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ with no convergent subsequence. Let A be the set of points in this sequence. We first claim that A is a closed subset with the discrete topology.

To see that A is closed, let $y \notin A$. Suppose that for all $\epsilon > 0$ there is an element of A contained in $B(y, \epsilon)$. Then we may inductively dene a subsequence converging to y by letting x_{n_i} be a point in $B(y, \epsilon)$ for $\epsilon = d(y, x_{n_{i-1}})/2$. Since there is no subsequence converging to y, we see that there is some $\epsilon > 0$ such that $B(y, \epsilon)$ is disjoint from A. Thus A is closed.

To see that A has the discrete topology, choose $x \in A$. Suppose that for all $\epsilon > 0$ there is an element of $A \setminus \{x\}$ contained in $B(x, \epsilon)$. Then we may inductively dene a subsequence converging to x by letting x_{n_i} be a point in $B(x, \epsilon) \setminus \{x\}$ for $\epsilon = d(y, x_{n_{i-1}})/2$. Since there is no subsequence converging to x, we see that there is some $\epsilon > 0$ such that $B(x, \epsilon) - \{x\}$ is disjoint from A. Thus A has the discrete topology.

Since $\{x_n\}_{n\in\mathbb{N}}$ has no convergent subsequence, A must have innitely many elements. Thus we can choose a surjection $f : A \to \mathbb{Z}$. Since A is discrete, f is continuous. By the Tietze extension theorem, f can be extended to a (necessarily unbounded) continuous function $g : X \to \mathbb{R}$. As this contradicts the hypotheses, it follows that X is compact.