Marta Lewicka, Math 2301, Fall 2019

Homework 1 – due Mon Sept 16

1. Check if the following linear spaces E are normed by the given function $\|\cdot\| : E \longrightarrow \mathbf{R}$. Are they Banach spaces?

(i) $E = \mathbf{R}^n$, $\|x\| = \left(\sum_{i=1}^n |x_i|^r\right)^{1/r}$, where $r \geq 1$ is a given number.

(ii) $E = l_2$, which are all sequences $x = \{x_i\}_{i=1}^\infty$ of real numbers that are square-summable: $\sum_{i=1}^\infty |x_i|^2 < \infty$. $\|x\| = \left(\sum_{i=1}^\infty |x_i|^2\right)^{1/2}$.

(iii) $E = l_2$ as in (ii), but: $\|f\| = \sup_{i\geq 1} |x_i|$.

2. The same question as in problem 1.

(i) $E = B(X)$, which are all bounded functions f from a set X to $\mathbf{R}$. $\|f\| = \sup\{|f(x)|;\ x \in X\}$.

(ii) $E = l_\infty$, which are all bounded sequences $x = \{x_n\}_{n=1}^\infty$ of real numbers. $\|x\| = \sup_{n\geq 1} |x_n|$.

(iii) $E = C([-1,1]) \cap C^1((-1,1))$, which are all continuous real functions f on the interval $[-1,1]$, that are continuously differentiable in $(-1,1)$. $\|f\| = \sup\{|f(x)|;\ x \in [-1,1]\}$.

3. Let E be a linear space and let $\|\cdot\|$ and $\|\cdot\|_1$ be two norms on E.

(i) Assume that $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent. Prove that $(E, \|\cdot\|)$ is a Banach space iff $(E, \|\cdot\|_1)$ is a Banach space.

(ii) Give an example of E, $\|\cdot\|$ and $\|\cdot\|_1$ so that $(E, \|\cdot\|)$ is a Banach space but $(E, \|\cdot\|_1)$ is not.

(iii) Prove that the function $\|\cdot\| : E \longrightarrow \mathbf{R}$ is continuous on $(E, \|\cdot\|)$. Must it be continuous on $(E, \|\cdot\|_1)$ as well?

4. Let E be a normed space and let $T \in E^*$. Does there necessarily exist $x \in E$ such that $\|x\| = 1$ and $|T(x)| = \|T\|$?

5. (i) Let E be a normed space and E_0 its linear subspace. Let $T_0 : E_0 \longrightarrow \mathbf{R}$ be a given linear functional. Show that there exists a linear functional $T : E \longrightarrow \mathbf{R}$ such that $T_{|E_0} = T_0$.

(ii) Let E be an infinitely dimensional normed space (which means that there exists an infinite sequence of its elements, which are linearly independent). Use part (i) to prove that there exists a discontinuous linear functional $T : E \longrightarrow \mathbf{R}$.