Homework 1 – due Mon Sept 16

1. Check if the following linear spaces $E$ are normed by the given function $\| \cdot \| : E \to \mathbb{R}$. Are they Banach spaces?
   (i) $E = \mathbb{R}^n$, $\| x \| = (\sum_{i=1}^{n} |x_i|^r)^{1/r}$, where $r \geq 1$ is a given number.
   (ii) $E = l_2$, which are all sequences $x = \{x_i\}_{i=1}^{\infty}$ of real numbers that are square-summable: $\sum_{i=1}^{\infty} |x_i|^2 < \infty$. $\| x \| = (\sum_{i=1}^{\infty} |x_i|^2)^{1/2}$.
   (iii) $E = l_2$ as in (ii), but: $\| f \| = \sup_{i \geq 1} |x_i|$.

2. The same question as in problem 1.
   (i) $E = B(X)$, which are all bounded functions $f$ from a set $X$ to $\mathbb{R}$. $\| f \| = \sup \{ |f(x)| : x \in X \}$.
   (ii) $E = l_\infty$, which are all bounded sequences $x = \{x_n\}_{n=1}^{\infty}$ of real numbers. $\| x \| = \sup_{n \geq 1} |x_n|$.
   (iii) $E = C([-1,1]) \cap C^1((-1,1))$, which are all continuous real functions $f$ on the interval $[-1,1]$, that are continuously differentiable in $(-1,1)$. $\| f \| = \sup \{ |f(x)| : x \in [-1,1] \}$.

3. Let $E$ be a linear space and let $\| \cdot \|$ and $\| \cdot \|_1$ be two norms on $E$.
   (i) Assume that $\| \cdot \|$ and $\| \cdot \|_1$ are equivalent. Prove that $(E, \| \cdot \|)$ is a Banach space iff $(E, \| \cdot \|_1)$ is a Banach space.
   (ii) Give an example of $E$, $\| \cdot \|$ and $\| \cdot \|_1$ so that $(E, \| \cdot \|)$ is a Banach space but $(E, \| \cdot \|_1)$ is not.
   (iii) Prove that the function $\| \cdot \| : E \to \mathbb{R}$ is continuous on $(E, \| \cdot \|)$. Must it be continuous on $(E, \| \cdot \|_1)$ as well?

4. Let $E$ be a normed space and let $T \in E^*$. Does there necessarily exist $x \in E$ such that $\| x \| = 1$ and $|T(x)| = \| T \|$?

5. (i) Let $E$ be a normed space and $E_0$ its linear subspace. Let $T_0 : E_0 \to \mathbb{R}$ be a given linear functional. Show that there exists a linear functional $T : E \to \mathbb{R}$ such that $T|_{E_0} = T_0$.
   (ii) Let $E$ be an infinitely dimensional normed space (which means that there exists an infinite sequence of its elements, which are linearly independent). Use part (i) to prove that there exists a discontinuous linear functional $T : E \to \mathbb{R}$.