Marta Lewicka, Math 2301, Fall 2019

Homework 10 – due Fri Dec 6

1. Let $f : [0,1] \longrightarrow \mathbf{R}$ be a continuous function. For each $c \in \mathbf{R}$ let $n(c)$ denote the number of solutions of the equation $f(x) = c$. Prove that the function $n : \mathbf{R} \longrightarrow \bar{\mathbf{R}}_+$ is Lebesgue measurable.

2. Prove that every (Lebesgue) measurable function $f : [0,1] \longrightarrow \mathbf{R}$ is a limit almost everywhere of a sequence $\{f_n\}$ of continuous functions. Is it always possible to choose this sequence to be monotone?

3. Let U be a bounded open subset of $\mathbf{R}^n$ and let $f : (a,b) \times U \longrightarrow \mathbf{R}$ be a continuous function such that for each $(t,x) \in (a,b) \times U$ the partial derivative $\partial f/\partial t\ (t,x)$ exists and satisfies: $\|\partial f/\partial t\ (t,x)\| \leq g(x)$, for some integrable function $g : U \longrightarrow \mathbf{R}$. Define the function: $F(t) := \int_U f(t,\cdot)\ \mathrm{d}\mu_n$. Prove that F is differentiable and that:

$$F'(t) = \int_U \frac{\partial f}{\partial t}(t,\cdot)\ \mathrm{d}\mu_n.$$

4. Prove that $\mathcal{L}_{n+m}$ is the smallest σ-algebra of subsets of $\mathbf{R}^{n+m}$, containing the product σ-algebra $\mathcal{L}_n \otimes \mathcal{L}_m$ and all sets of zero outer (Lebesgue) measure.

5. Let $(X, \mathcal{M}, \mu)$ be a finite measure space, so that $\mu(X) < \infty$. We say that a sequence of real-valued, integrable functions f_n on X is *uniformly integrable*, if $\sup_n\{\int_X |f_n|\ \mathrm{d}\mu\} < \infty$ and:

$$\forall \epsilon > 0 \quad \exists \delta > 0 \qquad \mu(A) < \delta \implies \forall n \quad \int_A |f_n|\ \mathrm{d}\mu < \epsilon.$$

Prove that a sequence f_n satisfies $\int_X |f_n - f|\ \mathrm{d}\mu \to 0$ if and only if both f_n converges to f in measure and the f_n are uniformly integrable.