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Homework 11 – due Wed Feb 12

1. Let E be a normed space. Let $\mathcal{A}_0$ be the collection of sets $U \subset E^*$ of the form:

$$U = \{T \in E^*;\ |T(x_i)| < \epsilon_i,\ i : 1 \dots n\} = \bigcap_{i=1}^n J(x_i)^{-1}(-\epsilon_i, \epsilon_i)$$

for some $n \geq 1$ and $\epsilon_i > 0$, $x_i \in E$, $i : 1 \dots n$. For a given $T_0 \in E^*$, define the family of translated sets U:

$$\mathcal{A}_{T_0} = \{T_0 + U;\ U \in \mathcal{A}_0\}.$$

(i) Prove that the weak * topology on E^* is composed of E^* and (arbitrary) unions of sets from the family: $\bigcup_{T_0 \in E^*} \mathcal{A}_{T_0}$.

(ii) Prove that $\mathcal{A}_{T_0}$ is a basis of open neighbourhoods of T_0 in the weak * topology.

Here, $J(x_i) \in E^{**}$ is the operator of evaluation on x_i.

2. Based on the first separation theorem from class, prove the following geometric form of the Hahn-Banach theorem. Let E be a normed space and let A, B be two nonempty, convex, disjoint subsets of E. Assume that one of the following conditions hold:

(i) at least one of the sets A, B is open,

(ii) both sets A, B are closed and at least one of them is compact.

Then there exists $T \in E^* \setminus \{0\}$ and $\alpha \in \mathbf{R}$ such that:

$$T(x) \leq \alpha \leq T(y) \qquad \forall x \in A \quad \forall y \in B.$$

3. Using the result from problem 2, prove the following distance formula. Let E_0 be a linear subspace of a normed space E. For $x_0 \in E$, define the distance:

$$d(x_0, E_0) = \inf\{||x_0 - x||;\ x \in E_0\}.$$

Define the set of functionals anihilating E_0:

$$E_0^\perp := \{T \in E^*;\ T_{|E_0} = 0\}.$$

Then:

$$d(x_0, E_0) = \sup\{|T(x_0)|;\ T \in E_0^\perp \text{ and } ||T|| \leq 1\}.$$

Notice that when $E_0 = \{0\}$ then we have known this result before.

4. Prove the following geometrical form of the Hahn-Banach theorem. Let E be a normed space and let W be its open, convex, nonempty subset. Let F be a linear subspace of E, such that $W \cap F = \emptyset$. Then there exists a hyperspace H of E (that is, $H = T^{-1}(0)$ for some linear, nonzero functional $T : E \longrightarrow \mathbf{R}$) which contains F and such that $W \cap H = \emptyset$. Moreover, H with these properties is automaticaly closed.

[Hint: First observe that the theorem is valid for $E = \mathbf{R}^2$. In the general setting, use Zorn's lemma to construct H.]

5. Let E be a normed space. Prove that if $\phi \in E^{**} \setminus J(E)$ then ϕ is not continuous with respect to the weak* topology on E^*. Deduce that the weak and the weak* topology on E^* coincide iff E is reflexive.

[Hint: Prove first that if $\phi : E^* \longrightarrow \mathbf{R}$ is linear and continuous with respect to the weak* topology (on E^*), then there exist $x_1, x_2 \dots, x_n \in E$ such that:

$$\bigcap_{i=1}^n Ker(J(x_i)) \subset Ker(\phi).$$

Then prove that necessarily ϕ must be a linear combination of $J(x_i)$.]