Homework 11 – due Wed Feb 12

1. Let $E$ be a normed space. Let $A_0$ be the collection of sets $U \subset E^*$ of the form:

$$U = \{ T \in E^*; \ |T(x_i)| < \epsilon_i, \ i = 1 \ldots n \} = \bigcap_{i=1}^{n} J(x_i)^{-1}(-\epsilon_i, \epsilon_i)$$

for some $n \geq 1$ and $\epsilon_i > 0$, $x_i \in E$, $i = 1 \ldots n$. For a given $T_0 \in E^*$, define the family of translated sets $U$:

$$A_{T_0} = \{ T_0 + U; \ U \in A_0 \}.$$ 

(i) Prove that the weak * topology on $E^*$ is composed of $E^*$ and (arbitrary) unions of sets from the family: $\bigcup_{T_0 \in E^*} A_{T_0}$.

(ii) Prove that $A_{T_0}$ is a basis of open neighbourhoods of $T_0$ in the weak * topology.

Here, $J(x_i) \in E^{**}$ is the operator of evaluation on $x_i$.

2. Based on the first separation theorem from class, prove the following geometric form of the Hahn-Banach theorem. Let $E$ be a normed space and let $A, B$ be two nonempty, convex, disjoint subsets of $E$. Assume that one of the following conditions hold:

(i) at least one of the sets $A, B$ is open,

(ii) both sets $A, B$ are closed and at least one of them is compact.

Then there exists $T \in E^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that:

$$T(x) \leq \alpha \leq T(y) \quad \forall x \in A \quad \forall y \in B.$$ 

3. Using the result from problem 2, prove the following distance formula. Let $E_0$ be a linear subspace of a normed space $E$. For $x_0 \in E$, define the distance:

$$d(x_0, E_0) = \inf\{|||x_0 - x||; \ x \in E_0\}.$$ 

Define the set of functionals anihilating $E_0$:

$$E^*_0 := \{ T \in E^*; \ T|_{E_0} = 0 \}.$$ 

Then:

$$d(x_0, E_0) = \sup\{|T(x_0)|; \ T \in E^*_0 \ \text{and} \ ||T|| \leq 1\}.$$ 

Notice that when $E_0 = \{0\}$ then we have known this result before.

4. Prove the following geometrical form of the Hahn-Banach theorem. Let $E$ be a normed space and let $W$ be its open, convex, nonempty subset. Let $F$ be a linear subspace of $E$, such that $W \cap F = \emptyset$. Then there exists a hyperspace $H$ of $E$ (that is, $H = T^{-1}(0)$ for some linear, nonzero functional $T : E \to \mathbb{R}$) which contains $F$ and such that $W \cap H = \emptyset$. Moreover, $H$ with these properties is automatically closed.

[Hint: First observe that the theorem is valid for $E = \mathbb{R}^2$. In the general setting, use Zorn’s lemma to construct $H$.]

5. Let $E$ be a normed space. Prove that if $\phi \in E^{**} \setminus J(E)$ then $\phi$ is not continuous with respect to the weak* topology on $E^*$. Deduce that the weak and the weak* topology on $E^*$ coincide iff $E$ is reflexive.

[Hint: Prove first that if $\phi : E^* \to \mathbb{R}$ is linear and continuous with respect to the weak* topology (on $E^*$), then there exist $x_1, x_2, \ldots, x_n \in E$ such that:

$$\cap_{i=1}^{n} \ker(J(x_i)) \subset \ker(\phi).$$

Then prove that necessarily $\phi$ must be a linear combination of $J(x_i)$.]