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Homework 12 – due Wed Feb 26

1. Prove Schur's Lemma: the weak and strong convergence in l_1 coincide.
The proof may be divided in the following steps:

(i) If the lemma was not correct, we would have a sequence $\{\{x_n^k\}_{n=1}^\infty\}_{k=1}^\infty$ converging weakly to 0 in l_1 (as $k \to 0$) and such that:
$$\|\{x_n^k\}_{n=1}^\infty\|_{l_1} \geq 1 \qquad \forall k.$$

(ii) One could find then an increasing sequence of natural numbers $\{n_k\}_{k=1}^\infty$ such that (without loss of generality):
$$\sum_{n=n_k}^{n_{k+1}-1} |x_n^k| > \frac{3}{4}\|\{x_n^k\}_{n=1}^\infty\|_{l_1} \qquad \forall k.$$

(iii) The above contradicts the weak convergence of our sequence to 0.

2. Prove that l_2 is a Hilbert space, and that the following spaces are not Hilbert: l_1, l_∞, $\mathcal{C}(K)$ (space of continuous functions on some compact subset metric space, with the sup norm).

3. Let E be a Banach space and let $F \neq E$ be its closed subspace.

(i) Prove that for every $\epsilon > 0$ there exists $x_\epsilon \in E$ of norm 1 and such that $\text{dist}(x_\epsilon, F) \geq (1-\epsilon)$. (We say that x_ϵ is ϵ-perpendicular to F).

(ii) Using (i), prove that the closed unit ball in E is compact (in the strong topology) iff E has finite dimension.

4. Prove or disprove the following statement. Every Banach space in which the parallelogram identity holds is a Hilbert space (in the sence that it admits a scalar product which induces its norm).

5. Use the following outline to prove that the unit ball $\overline{B}_{E^*}$ in E^* (E is a Banach space) with weak * topology is metrizable iff E is separable.
Proof of separability $\implies$ metrizability:

(i) Find a sequence x_n of elements in $\overline{B}_E$, dense in this ball, and define:
$$d(T,S) := \sum_{n=1}^\infty \frac{1}{2^n}|(T-S)(x_n)| \qquad \forall T, S \in \overline{B}_{E^*}.$$
Check that d is a metric on $\overline{B}_{E^*}$.

(ii) Take a basic weak * open neighbourhood U of T in $\overline{B}_{E^*}$, given through evaluations at points $y_1, \dots y_k$ in $\overline{B}_E$. Approximate each y_i by a x_{n_i} and choose r much smaller than each 2^{-n_i}. The open ball centered at T and of radius r with respect to the metric d should then be contained in U.

(iii) Conversly, think of an open ball centered at T and of radius $r > 0$ with respect to the metric d. Construct its subset which is basic weak * open neighbourhood of T in $\overline{B}_{E^*}$. The convergence of the series in the definition of d is a hint.

Proof of mertizability $\implies$ separability:

(iv) Consider a decreasing sequence of open balls B_n centered at 0 and of radii say $1/n$ with respect to the metric. Each B_n contains a basic weak * open neighbourhood of 0 in $\overline{B}_{E^*}$, given through evaluations at a finite collection of points $A_n \subset \overline{B}_E$. Take $D = \bigcup A_n$. The subspace $F = \text{span}(D)$ is dense in E because $F^\perp = \{0\}$.