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Homework 13 – due Wed March 25

1. Prove the following generalization of the Holder inequality. Let $1 \leq p_i \leq \infty$, where $i : 1..k$ satisfy $\sum_{i=1}^k 1/p_i = 1/p$ for some $p \in [1, \infty]$. If $f_i \in L^{p_i}(\Omega)$, $i : 1 \ldots k$, then the product $f_1 ... f_n$ belongs to the space $L^p(\Omega)$ and:

$$||f_1 ... f_n||_{L^p} \leq ||f_1||_{L^{p_1}} ... ||f_n||_{L^{p_n}}.$$

2. Prove the following interpolation inequality. If $f \in L^p(\Omega) \cap L^q(\Omega)$, for some $1 \leq p \leq q \leq \infty$, then $f \in L^r(\Omega)$ for any $r \in [p, q]$ and there holds:

$$||f||_{L^r} \leq ||f||^{\alpha}_{L^p} ||f||^{1-\alpha}_{L^q}, \qquad \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}, \ \alpha \in [0, 1].$$

3. Suppose that $\Omega \subset \mathbf{R}^n$ has finite measure and let $1 \leq p, q \leq \infty$.

(i) Prove that if $f \in L^q(\Omega)$ and $p \leq q$, then $f \in L^p(\Omega)$ and:

$$||f||_{L^p} \leq (\mu(\Omega))^{1/p - 1/q} ||f||_{L^q}.$$

(ii) Prove that for any pair of distinct exponents $p \neq q$ we have: $L^p(\mathbf{R}) \not\subset L^q(\mathbf{R})$. Hint: consider the function:

$$f(x) = \frac{1}{|x|^{1/2} \sqrt{1 + \log^2 |x|}}.$$

4. Let $f \in L^p(\mathbf{R}^n)$, $g \in L^q(\mathbf{R}^n)$, $h \in L^r(\mathbf{R}^n)$, $p, q, r \in [1, \infty]$.

(i) Prove that if $1/p + 1/q - 1 = 1/r \geq 0$, then $f * g \in L^r(\mathbf{R}^n)$ and $\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$.

(ii) Prove the following version of Young's Theorem. If $1/p + 1/q + 1/r = 2$, then:

$$\left| \int_{\mathbf{R}^n} (f * g) h \right| \leq \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^r}.$$

5. Prove that every Banach space E is linearly isometric to a closed subspace of $\mathcal{C}(X)$, where X is a compact topological space. When E is separable, prove that X may be taken as the interval $[0, 1]$.

[Hint: If X is a compact metric space which can be seen as a convex subset of some linear space, then show that there exists a continuous function f from $[0, 1]$ onto X.]