1. Prove the following generalization of the Holder inequality. Let $1 \leq p_i \leq \infty$, where $i : 1..k$ satisfy $\sum_{i=1}^{k} \frac{1}{p_i} = 1/p$ for some $p \in [1, \infty]$. If $f_i \in L^{p_i}(\Omega), \ i : 1..k$, then the product $f_1...f_n$ belongs to the space $L^p(\Omega)$ and:
$$
||f_1...f_n||_{L^p} \leq ||f_1||_{L^{p_1}} ... ||f_n||_{L^{p_n}}.
$$

2. Prove the following interpolation inequality. If $f \in L^p(\Omega) \cap L^q(\Omega)$, for some $1 \leq p \leq q \leq \infty$, then $f \in L^r(\Omega)$ for any $r \in [p,q]$ and there holds:
$$
||f||_{L^r} \leq ||f||_{L^p}^{1-\alpha} ||f||_{L^q}^{\alpha}, \ \ 1 = \frac{\alpha}{p} + \frac{1-\alpha}{q}, \ \ \alpha \in [0,1].
$$

3. Suppose that $\Omega \subset \mathbb{R}^n$ has finite measure and let $1 \leq p,q \leq \infty$.
(i) Prove that if $f \in L^q(\Omega)$ and $p \leq q$, then $f \in L^p(\Omega)$ and:
$$
||f||_{L^p} \leq \frac{1}{\mu(\Omega)^{1/p-1/q}} ||f||_{L^q}.
$$
(ii) Prove that for any pair of distinct exponents $p \neq q$ we have: $L^p(\mathbb{R}) \not\subset L^q(\mathbb{R})$. Hint: consider the function:
$$
f(x) = \frac{1}{|x|^{1/2} \sqrt{1 + \log^2 |x|}}.
$$

4. Let $f \in L^p(\mathbb{R}^n), \ g \in L^q(\mathbb{R}^n), \ h \in L^r(\mathbb{R}^n), \ p,q,r \in [1,\infty]$.
(i) Prove that if $1/p + 1/q - 1 = 1/r \geq 0$, then $f * g \in L^r(\mathbb{R}^n)$ and $||f * g||_{L^r} \leq ||f||_{L^p} ||g||_{L^q}$.
(ii) Prove the following version of Young’s Theorem. If $1/p + 1/q + 1/r = 2$, then:
$$
\left| \int_{\mathbb{R}^n} (f * g)h \right| \leq ||f||_{L^p} ||g||_{L^q} ||h||_{L^r}.
$$

5. Prove that every Banach space $E$ is linearly isometric to a closed subspace of $C(X)$, where $X$ is a compact topological space. When $E$ is separable, prove that $X$ may be taken as the interval $[0,1]$.
[Hint: If $X$ is a compact metric space which can be seen as a convex subset of some linear space, then show that there exists a continuous function $f$ from $[0,1]$ onto $X$.]