1. Let $1 < p < \infty$. For every $f \in L^p((0,\infty))$, define:
   \[ F(x) = \frac{1}{x} \int_0^x f(t) \, dt \quad \forall x \in (0,\infty). \]
   Prove that $F \in L^p((0,\infty))$ and:
   \[ \|F\|_{L^p} \leq \frac{p}{p-1} \|f\|_{L^p}. \]
   [Hint: Assume first that $f$ is nonnegative and compactly supported in $(0,\infty)$. Integrate by parts and notice that $xF'(x) = f(x) - F(x).$]

2. Using Jensen's inequality, prove:
   (i) Holder’s inequality [Hint: Use $\phi(x) = x^p$ and consider first the case of $f, g$ nonnegative and such that $\|g\|_{L^q} = 1.$]
   (ii) Minkowski’s inequality (that is, the triangle inequality for the norm in $L^p$ [Hint: Use $\phi(x) = (1 - x^{1/p})^p$ and assume first that $f, g$ are nonnegative, $f \leq 1$ and $\|f + g\|_{L^p} = 1.$]

3. For which functions $f \in L^p$ and $g \in L^{p'}$ we have “equality” in Holder’s inequality?:
   \[ \|fg\|_{L^1} = \|f\|_{L^p} \|g\|_{L^{p'}}. \]

4. Prove the converse of the Riesz - Frechet - Kolmogorov theorem. Assume that a subset $\mathcal{F}$ of $L^p(\mathbb{R}^n)$ has compact closure. (We assume that $1 < p < \infty.$) Then there holds:
   (i) $\exists C > 0 \quad \forall f \in \mathcal{F} \quad \|f\|_{L^p(\mathbb{R}^n)} \leq C,$
   (ii) $\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall h \in B(0, \delta) \quad \forall f \in \mathcal{F} \quad \|h \ast f - f\|_{L^p(\mathbb{R}^n)} < \epsilon,$
   (iii) $\forall \epsilon > 0 \quad \exists \Omega \in \mathcal{L}_n$, bounded $\forall f \in \mathcal{F} \quad \|f\|_{L^p(\mathbb{R}^n \setminus \Omega)} < \epsilon.$

5. Let $\Omega \subset \mathbb{R}^n$ be open, bounded and of (Lebesgue) measure 1. Let $f \in L^1(\Omega)$. Prove that:
   \[ \lim_{p \to 0} \left( \int_\Omega |f|^p \right)^{1/p} = \exp \left( \int_\Omega \ln |f| \right). \]
   [Hint: Use Jensen’s inequality to prove $\leq$. For the converse inequality, notice that $\ln |x| \leq |x| - 1$ and $\lim_{p \to 0} (|x|^p - 1)/p = \ln |x|.$]