Marta Lewicka, Math 2302, Spring 2020

Homework 14 – due Wed April 1

1. Let $1 < p < \infty$. For every $f \in L^p((0,\infty))$, define:

$$F(x) = \frac{1}{x}\int_0^x f(t)\ \mathrm{d}t \qquad \forall x \in (0,\infty).$$

Prove that $F \in L^p((0,\infty))$ and:

$$\|F\|_{L^p} \leq \frac{p}{p-1}\|f\|_{L^p}.$$

[Hint: Assume first that f is nonnegative and compactly supported in $(0,\infty)$. Integrate by parts and notice that $xF'(x) = f(x) - F(x)$.]

2. Using Jensen's inequality, prove:

(i) Holder's inequality [Hint: Use $\phi(x) = x^p$ and consider first the case of f, g nonnegative and such that $\|g\|_{L^q} = 1$.]

(ii) Minkowski's inequality (that is, the triangle inequality for the norm in L^p [Hint: Use $\phi(x) = (1 - x^{1/p})^p$ and assume first that f, g are nonnegative, $f \leq 1$ and $\|f+g\|_{L^p} = 1$.]

3. For which functions $f \in L^p$ and $g \in L^{p'}$ we have "equality" in Holder's inequality?:

$$\|fg\|_{L^1} = \|f\|_{L^p}\|g\|_{L^{p'}}$$

4. Prove the converse of the Riesz - Frechet - Kolmogorov theorem. Assume that a subset $\mathcal{F}$ of $L^p(\mathbf{R}^n)$ has compact closure. (We assume that $1 < p < \infty$.) Then there holds:

(i) $\exists C > 0 \quad \forall f \in \mathcal{F} \qquad \|f\|_{L^p(\mathbf{R}^n)} \leq C$,

(ii) $\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall h \in B(0,\delta) \quad \forall f \in \mathcal{F} \qquad \|\tau_h f - f\|_{L^p(\mathbf{R}^n)} < \epsilon$,

(iii) $\forall \epsilon > 0 \quad \exists \Omega \in \mathcal{L}_n$, bounded $\quad \forall f \in \mathcal{F} \qquad \|f\|_{L^p(\mathbf{R}^n \setminus \Omega)} < \epsilon$.

5. Let $\Omega \subset \mathbf{R}^n$ be open, bounded and of (Lebesgue) measure 1. Let $f \in L^1(\Omega)$. Prove that:

$$\lim_{p\to 0}\left(\int_\Omega |f|^p\right)^{1/p} = \exp\left(\int_\Omega \ln|f|\right).$$

[Hint: Use Jensen's inequality to prove $\leq$. For the converse inequality, notice that $\ln|x| \leq |x| - 1$ and $\lim_{p\to 0}(|x|^p - 1)/p = \ln|x|$.]