Marta Lewicka, Math 2302, Spring 2020

Homework 15 – due Mon April 20

1. Let $p \in [1,\infty]$, $u \in W^{1,p}(\Omega)$, and let $f \in \mathcal{C}^\infty(\mathbf{R})$ be such that $f(0) = 0$ and $f' \in L^\infty$. Prove that: $f \circ u \in W^{1,p}(\Omega)$ and $\nabla(f \circ u) = (f' \circ u)\nabla u$.

2. Let $p \in [1,\infty]$, $u \in W^{1,p}(\Omega)$ and let $h : \tilde{\Omega} \longrightarrow \Omega$ be a diffeomorphism of class $\mathcal{C}^\infty$ between the open sets $\tilde{\Omega}$ and Ω. Assume that both $\det\nabla h$ and $\det\nabla(h^{-1})$ are in L^∞. Prove that $u \circ h \in W^{1,p}(\tilde{\Omega})$ and $\nabla(u \circ h) = [(\nabla u) \circ h]\nabla h$.

3. Let $p \in [2,\infty)$. We define:

$$W_0^{1,p}(\Omega) = closure_{W^{1,p}(\Omega)}\mathcal{C}_c^\infty(\Omega).$$

Prove the following interpolation inequality:

$$\forall u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \qquad \|\nabla u\|_{L^p} \leq C\|u\|_{L^p}^{1/2}\|\nabla^2 u\|_{L^p}^{1/2}$$

where the constant C depands only on Ω and p.

4. Let $p \in [1,\infty]$. For $u \in W^{1,p}(\Omega)$, define $u^+ = \max(u, 0)$.

(i) Prove that $u^+ \in W^{1,p}(\Omega)$ and that:

$$\nabla(u^+) = \begin{cases} \nabla u & \text{a.e. in } \{x;\ u(x) > 0\} \\ 0 & \text{a.e. in } \{x;\ u(x) \leq 0\}. \end{cases}$$

(ii) Given $c \in \mathbf{R}$, show that $\nabla u = 0$ almost everywhere in the (measurable) set $\{x;\ u(x) = c\}$.

(iii) Prove that if the sequence $\{u_n\}$ converges to u in $W^{1,p}(\Omega)$, then also $\{u_n^+\}$ converges to u^+.

5. Let $\Omega \subset \mathbf{R}^n$ be open, bounded and of class $\mathcal{C}^1$. Let $u \in W^{k,p}(\Omega)$. Using the established facts for $W^{1,p}(\Omega)$, deduce the following statement. If $k < n/p$ then $u \in L^q(\Omega)$, where $1/q = 1/p - k/n$ and for some constant C depending only on k, p, n and Ω one has:

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)}.$$