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Homework 2 – due Fri Sept 20

1. Let s be the linear space of all sequences of real numbers.

(i) Prove that the function $d : s \times s \longrightarrow \mathbf{R}$:

$$d(\{x_i\},\{y_i\}) := \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{|x_i - y_i|}{1 + |x_i - y_i|}$$

is a metric on s, and that the metric space (s,d) is complete.

(ii) Prove that every open neighbourhood of 0 in s contains the whole line $\{\alpha x;\ \alpha \in \mathbf{R}\}$, for some $x \in s \setminus \{0\}$.

(iii) Deduce from (ii) that there is no norm in s, which would make it a Banach space, topologically equivalent to (s,d) (which means, with the convergence of sequences in this norm the same as the convergence in the metric space (s,d)).

2. Prove that for every $x \neq y$ in a normed space E, there exists $T \in E^*$ such that $T(x) < T(y)$.

3. Find the norms of the following linear functionals on $\mathcal{C}[a,b]$ (with the norm of the uniform convergence $\|f\| = \max\{|f(x)|;\ x \in [a,b]\}$):

(i) $T(f) := \int_a^b f(x)\ \mathrm{d}x$,

(ii) $T_g(f) := \int_a^b f(x)g(x)\ \mathrm{d}x$, where g is a fixed element of $\mathcal{C}[a,b]$,

(iii) $T(f) := \sum_{i=1}^n \lambda_i \cdot f(x_i)$, where $x_1 ... x_n \in [a,b]$ and $\lambda_1 ... \lambda_n \in \mathbf{R}$ are given parameters.

4. We say that a normed space E is strictly convex iff:

$$\forall x \neq y \in E \quad \forall t \in (0,1) \quad \|x\| = \|y\| = 1 \implies \|tx + (1-t)y\| < 1.$$

Prove that if E^* is strictly convex then for every $x_0 \in E$ there exists EXACTLY ONE functional $T \in E^*$ such that $\|T\| = \|x_0\|$ and $T(x_0) = \|x_0\|^2$.

5. In the linear space c_0 of all sequences of real numbers converging to 0, consider the sequence $\{e^i\}_{i=1}^{\infty}$, where $e^i = (0,0,0,\ldots 1, 0 \ldots)$ is such that 1 is only on the i-th place in the sequence e^i.

(i) Prove that $\{e^i\}_{i=1}^{\infty}$ is a Schauder basis in l_2.

(ii) Prove that $\{e^i\}_{i=1}^{\infty}$ is a Schauder basis in c_0. Recall that c_0 is normed by the l_∞ norm.