Marta Lewicka, Math 2301, Fall 2019

Homework 3 – due Fri Sept 27

1. Prove that every metric space X is paracompact (i.e. every open covering of X admits an open neighbourhood-finte refinement).

2. Recall that a metric space Y is an extensor, if for every continuous function $f : A \longrightarrow Y$ defined on a closed subset A od a mteric space X, there exists a continuous function $\tilde{f} : X \longrightarrow Y$ such that $\tilde{f}(x) = f(x)$ for every $x \in A$.

Prove that:

(i) a space homeomorphic to an extensor is also an extensor,

(ii) a retract of an extensor is an extensor,

(iii) if Y_1 and Y_2 are extensors, then $Y_1 \times Y_2$ is an extensor.

3. Find the norms of the following linear functionals on $\mathcal{C}[-1, 1]$:

(i) $T(f) := \int_0^1 f(x)\ \mathrm{d}x$,

(ii) $T(f) := \int_{-1}^1 (\mathrm{sgn}\ x) f(x)\ \mathrm{d}x$,

(iii) $T(f) := \int_{-1}^1 f(x)\ \mathrm{d}x - f(0)$,

(iv) $T(f) := \frac{f(\epsilon)+f(-\epsilon)-2f(0)}{\epsilon^2}$,

(v) $T(f) := \sum_{n=1}^\infty \frac{(-1)^n}{n^2} f(1/n)$

4. Prove that the space $\mathcal{C}_b(\mathbb{R}^N)$ of bounded continuous functions on $\mathbb{R}^N$, with the supremum norm $\|\cdot\|_\infty$, is not separable.

Hint: Modify the proof of nonseparability of l_∞.

5. Let (X, d) be a metric space. Fix a reference point $x_0 \in X$ and let E be the vector space of all the Lipschitz continuous functions $f : X \longrightarrow \mathbf{R}$ such that $f(x_0) = 0$. Define $||f||$ to be the smallest Lipschitz constant of f, that is:

$$||f|| := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}.$$

Prove that $(E, ||\cdot||)$ is a Banach space.