Marta Lewicka, Math 2301, Fall 2019

Homework 4 – due Fri Oct 4

1. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $x_0 \in \mathbb{R}^n$.

(i) Prove that $f'(x_0)$ is the linear, automatically continuous, operator from $\mathbb{R}^n$ to $\mathbb{R}^m$ that is given by the matrix $A \in \mathbb{R}^{m\times n}$ whose components are partial derivatives: $A_{ij} = \frac{\partial f^i}{\partial x_j}(x_0)$. This is the Jacobi matrix.

(ii) Prove that if all partial derivatives of f exist and are continuous, then f is (strongly) differentiable at each $x_0 \in \mathbb{R}^n$.

2. Let E be a Banach space and let F be its finitely dimensional subspace. Show that there exists a closed subspace $G \subset E$ such that:

$$F + G = E \quad \text{and} \quad F \cap G = \{0\}.$$

[Hint: Use Hahn-Banach Theorem.]

3. (i) Let $(E, \|\cdot\|)$ be the Banach space defined in problem 5 of Homework 3. For each $x \in X$, let $T_x(f) = f(x)$ Prove that each $T_x \in E^*$ and that $\|T_x - T_y\| = d(x, y)$. Deduce that X is therefore isometric to a subset of E^*.

(ii) Let E be a linear normed space. Prove that it is isometric with a linear subspace of the Banach space $\mathcal{B}(B_{E^*}(1))$ of bounded real functions on the closed unit ball in the dual space E^*: $B_{E^*}(1) = \{T \in E^*;\ \|T\| \leq 1\}$.

4. Let (Y, d) be a complete metric space and let $f : B \longrightarrow Y$ be a contractive mapping with Lipschitz constant $\alpha < 1$. Here $B \subset Y$ is an open ball centered at some $y_0 \in Y$ and radius $r > 0$. Prove that if $d(f(y_0), y_0) < (1 - \alpha)r$ then f has a fixed point.

5. Let (X, d) be a complete metric space and $f : X \longrightarrow X$ a map such that for some $n > 1$ the composition of the function f with itself n times: $f^{(n)} : X \longrightarrow X$ is a contraction.

(i) Does f have to be continuous?

(ii) Prove that f has the unique fixed point in X.