Marta Lewicka, Math 2301, Fall 2019

Homework 5 – due Fri Oct 11

1. Let Λ be a metric space, E a Banach space, and let $F : \Lambda \times E \longrightarrow E$ be a function such that:

$$\exists \kappa < 1 \ \forall \lambda \in \Lambda \ \forall x, y \in E \quad \|F(\lambda, x) - F(\lambda, y)\| \leq \kappa \|x - y\|.$$

Prove that:

(i) for every $\lambda \in \Lambda$ there exists a unique $x(\lambda) \in E$ such that $x(\lambda) = F(\lambda, x(\lambda))$,

(ii) for every $\lambda \in \Lambda$, $y \in E$ one has:

$$\|x(\lambda) - F(\lambda, y)\| \leq \frac{\kappa}{1-\kappa} \|y - F(\lambda, y)\|,$$

$$\|y - x(\lambda)\| \leq \frac{1}{1-\kappa} \|y - F(\lambda, y)\|.$$

2. Let $f_1 : E \longrightarrow F_1$ and $f_2 : E \longrightarrow F_2$ be two differentiable mappings between Banach spaces E, F_1, F_2. Define $f : E \longrightarrow F_1 \times F_2$ by: $f(x) = (f_1(x), f_2(x))$. Prove that f is differentiable and find its derivative. (Here $F_1 \times F_2$ is the Banach space equipped with the norm $\|(y_1, y_2)\| := \|y_1\|_{F_1} + \|y_2\|_{F_2}$.)

3. Let E, F, G be normed spaces and let $\phi : E \times F \longrightarrow G$ be a bilinear map (i.e. such that both maps: $\phi(\cdot, y) : E \longrightarrow G$ and $\phi(x, \cdot) : F \longrightarrow G$ are linear, for every $x \in E$ and $y \in F$).

(i) Prove that ϕ is continuous iff it is bounded, that is:

$$\exists C > 0 \quad \forall x \in E \ \forall y \in F \qquad \|\phi(x, y)\| \leq C \cdot \|x\| \cdot \|y\|.$$

(ii) Let $\mathcal{L}(E, F; G)$ be the linear space of all continuous bilinear mappings ϕ, as above. Prove that it is a normed space, with the norm defined as:

$$\|\phi\| := \sup \{\|\phi(x, y)\|; \ \|x\| \leq 1, \ \|y\| \leq 1\}.$$

(iii) Prove that if G is Banach then $\mathcal{L}(E, F; G)$ is also Banach.

4. For $p \in (0, 1)$, define l_p and $|| \cdot ||_{l_p}$ by the standard formula. Show that $|| \cdot ||_{l_p}$ is not a norm.

5. Give an example of a discontinuous linear map between normed spaces, so that:

(i) its graph is closed and its target space is Banach,

(ii) its graph is closed and its domain space is Banach.