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Homework 6 – due Fri Oct 18

1. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of $\mathcal{C}^1$ maps from an open subset U of a Banach space E into a Banach space F. Assume that $\{f_n\}$ converge pointwise to a map $f : U \longrightarrow F$ and that the sequence of derivatives $\{f'_n\}$ converges uniformly to a mapping $g : U \longrightarrow \mathcal{L}(E, F)$. Prove that f is $\mathcal{C}^1$ and that $f' = g$.

2. Let $f \in \mathcal{C}^{k+1}(U, F)$ where U is an open subset of a Banach space E and F is another Banach space. Let $x_0 \in U$ and $v \in E$ be such that $x_0 + tv \in U$, for every $t \in [0, 1]$. Prove the Taylor's formula:

$$f(x_0 + v) = f(x_0) + \left(\sum_{i=1}^{k} \frac{1}{i!} D^i f(x_0)(v, \ldots, v) \right) + R_k(x_0, v),$$

where $\|R_k(x_0, v)\|/\|v\|^k \to 0$ as $v \to 0$.

[Hint: As in the proof of the mean value theorem and the symmetry of the second derivative, use the Hahn-Banach theorem to reduce the statement to the case of a real function of one variable.]

3. Let E be a Banach space. Show that the mapping $Inv : \mathcal{GL}(E, E) \longrightarrow \mathcal{GL}(E, E)$ given by $Inv(T) = T^{-1}$ is differentiable and find its derivative.

4. Let U be an open subset of a Banach space E. Given a function $g \in \mathcal{C}^1(U, \mathbf{R})$, define the mapping

$$S_g : \mathcal{C}([0, 1], U) \longrightarrow \mathbf{R}, \qquad S_g(f) = \int_0^1 g(f(s))\, \mathrm{d}s.$$

Show that S_g is $\mathcal{C}^1$ and find its derivative.

5. Let M be some σ-algebra of subsets of X, let N be some σ-algebra of subsets of Y. Given is a function $f : X \longrightarrow Y$. Which of the following families is a σ-algebra? Provide a proof or a counterexample.

a) $\{B \subset Y;\ f^{-1}(B) \in M\}$,
b) $\{f(A);\ A \in M\}$,
c) $\{A \subset X;\ f(A) \in N\}$,
d) $\{f^{-1}(B);\ B \in N\}$.