Marta Lewicka, Math 2301, Fall 2019

Homework 7 – due Fri Oct 25

1. (i) Let μ^* be an external measure (i.e. a measure generator) on X. Assume that for some $A \subset X$ there holds: $\mu^*(A) + \mu^*(X \setminus A) = \mu^*(X) < \infty$. Does A have to be measurable with respect to μ^*?
(ii) Give an example of an external measure which is not a measure.

2. Let $p, q \in (1, \infty)$. For which functions f the operator of multiplication by f is continuous from $L^p[0,1]$ to $L^q[0,1]$?

3. Prove that a linear functional on a normed space (with values in $\mathbf{R}$) is bounded iff its kernel is closed.

4. Let $f, \{f_n\}_{n=1}^\infty : X \longrightarrow \mathbf{R}$ be μ-measurable functions, for some measure μ on a σ-algebra $\mathcal{M} \subset 2^X$. We say that f_n *converges almost everywhere* to f if:

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for every point x outside some subset of measure 0.

We say that f_n *converges in measure* to f if for every $\epsilon > 0$:

$$\lim_{n\to\infty} \mu\left(x \in X;\ |f_n(x) - f(x)| \geq \epsilon\right) = 0.$$

(i) Prove that if μ is a finite measure then convergence almost everywhere implies convergence in measure.
(ii) In (i), can we omit the assumption of finiteness of μ?
(iii) In (i), is the converse implication true?

5. In the setting of problem 4:

(i) Prove that if f_n converges almost everywhere to some f and if f_n converges almost everywhere to another function g, then $f(x) = g(x)$ for every point x outside some subset of measure 0.
(ii) The same for convergence in measure.