1. Let $F : A \to \mathbb{R}$ be a continuous function on a compact set $A \subset \mathbb{R}^N$. Verify that, setting:

$$F(x) := \min_{y \in A} \left\{ F(y) + \frac{|x - y|}{\text{dist}(x, A)} - 1 \right\}$$

for all $x \in \mathbb{R}^N \setminus A$, defines a continuous extension of $F$ on $\mathbb{R}^N$.

(This construction is due to Hausdorff and it provides a proof of the Tietze extension theorem (it may be generalized to $A$ a closed subset of a metric space, $F$ continuous and bounded from below on $A$).]

2. Let $(X, \mathcal{M}, \mu)$ be a measure space. For every $A \subset X$, define:

$$\mu^*(A) := \inf \{ \mu(B); A \subset B, B \in \mathcal{M} \}.$$  

(i) Show that $\mu^*$ is a measure generator, coinciding with $\mu$ on $\mathcal{M}$ and such that it is 0 on every subset of a zero $\mu$-measure set.

(ii) Let $\mathcal{M}_c$ be the $\sigma$-algebra generated by $\mu^*$. Show that $\mathcal{M} \subset \mathcal{M}_c$.

(iii) Is the following characterisation true?:

$$\mathcal{M}_c = \{ A \in 2^X; \exists B \in \mathcal{M} \ A \subset B \text{ and } \mu(B) = \mu^*(A) \text{ and } \mu^*(B \setminus A) = 0 \}$$

3. Let $f : [a, b] \to \mathbb{R}$ be a given function.

(i) If $f$ is continuous, show that its graph is a set of (Lebesgue) measure 0 in $\mathbb{R}^2$.

(ii) What if $f$ is just a (possibly discontinuous) monotone function?

4. Prove that the following subsets of $[0, 1]$ are compact, of Lebesgue measure 0 and uncountable:

(i) the set $A$ containing all numbers which admit a binary representation $0, c_1 c_2 c_3 ...$ such that $c_n = 0$ for all $n$ odd,

(ii) the set $B$ of all numbers which admit a binary representation $0, c_1 c_2 c_3 ...$ such that for every $n$ there is: $c_n = 0$ or $c_{n+1} = 0$.

5. Show that the derivative of a differentiable function $f : (a, b) \to \mathbb{R}$ is a (Lebesgue) measurable function.