Marta Lewicka, Math 2301, Fall 2019

Homework 8 – due Fri Nov 8

1. Let $F : A \to \mathbf{R}$ be a continuous function on a compact set $A \subset \mathbf{R}^N$. Verify that, setting:

$$F(x) \doteq \min_{y\in A} \Big\{ F(y) + \frac{|x-y|}{\mathrm{dist}(x,A)} - 1 \Big\} \qquad \text{for all } \; x \in \mathbf{R}^N \setminus A,$$

defines a continuous extension of F on $\mathbf{R}^N$.

[This construction is due to Hausdorff and it provides a proof of the Tietze extension theorem (it may be generlized to A a closed subset of a metric space, F continuous and bounded from below on A).]

2. Let $(X, \mathcal{M}, \mu)$ be a measure space. For every $A \subset X$, define:

$$\mu^*(A) := \inf\{\mu(B); \; A \subset B, \; B \in \mathcal{M}\}.$$

(i) Show that μ^* is a measure generator, coinciding with μ on $\mathcal{M}$ and such that it is 0 on every subset of a zero μ-measure set.

(ii) Let $\mathcal{M}_c$ be the σ-algebra generated by μ^*. Show that $\mathcal{M} \subset \mathcal{M}_c$.

(iii) Is the following characterisation true?:

$$\mathcal{M}_c = \Big\{ A \in 2^X; \; \exists B \in \mathcal{M} \quad A \subset B \text{ and } \mu(B) = \mu^*(A) \text{ and } \mu^*(B \setminus A) = 0 \Big\}$$

3. Let $f : [a,b] \longrightarrow \mathbf{R}$ be a given function.

(i) If f is continuous, show that its graph is a set of (Lebesgue) measure 0 in $\mathbf{R}^2$.

(ii) What if f is just a (possibly discontinuous) monotone function?

4. Prove that the following subsets of $[0,1]$ are compact, of Lebesgue measure 0 and uncountable:

(i) the set A containing all numbers which admit a binary representation $0, c_1c_2c_3...$ such that $c_n = 0$ for all n odd,

(ii) the set B of all numbers which admit a binary representation $0, c_1c_2c_3...$ such that for every n there is: $c_n = 0$ or $c_{n+1} = 0$.

5. Show that the derivative of a differentiable function $f : (a,b) \longrightarrow \mathbf{R}$ is a (Lebesgue) measurable function.