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Homework 9 – due Fri Nov 22

1. Given an open box $A = (a_1, b_1) \times \ldots \times (a_n, b_n) \subset \mathbf{R}^n$ of finite, positive volume, construct a sequence ϕ_n of functions in $\mathcal{C}_c^\infty(\mathbf{R}^n, \mathbf{R})$ such that: $\phi_n \nearrow \mathbf{1}_A$ (the convergence is pointwise everywhere).

2. Prove that if $A \in \mathcal{L}_{n+m}$ has measure 0, then for almost every $x \in \mathbf{R}^n$ the set:

$$A_x := \{y \in \mathbf{R}^k;\ (x, y) \in A\}$$

is in $\mathcal{L}_m$ and has measure 0.

3. Using the result of problem 2 and the Fubini-Tonelli theorem for product measures deduce the following Fubini-Tonelli theorem for the Lebesgue integral.

Let $f : \mathbf{R}^{n+m} \longrightarrow \overline{\mathbf{R}}$ be a nonnegative (Lebesgue) measurable or a (Lebesgue) integrable function. Then:

(i) For almost every $x \in \mathbf{R}^n$ the function $y \mapsto f(x, y)$ is measurable and its integral $\int_{\mathbf{R}^m} f(x, \cdot)\ \mathrm{d}\mu_m$ is well defined.

(ii) The function

$$x \mapsto \begin{cases} \int_{\mathbf{R}^m} f(x, \cdot)\ \mathrm{d}\mu_m & \text{when defined} \\ 0 & \text{elsewhere} \end{cases}$$

is measurable and its integral over $\mathbf{R}^n$ is well defined.

(iii) One has the following formula:

$$\int_{\mathbf{R}^{n+m}} f\ \mathrm{d}\mu_{m+n} = \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^m} f\ \mathrm{d}\mu_m \right)\ \mathrm{d}\mu_n.$$

4. Let $f : [0, \pi] \longrightarrow \mathbf{R}$ be given by:

$$f_n(x) = n \frac{\sin x}{1 + n^2 \sin^2 x}.$$

For a given $\epsilon > 0$, find explicitly the Egoroff set E_ϵ on which the sequence f_n converges uniformly, and such that $\mu(E_\epsilon) > \pi - \epsilon$.

5. Prove that for every set $A \subset \mathbf{R}^n$ which is not of Lebesgue measure 0, there holds:

$$\forall c \in (0, 1) \quad \exists P \subset \mathbf{R}^n \qquad \mu^*(A \cap P) > c\mu^*(P).$$

(P is a closed box and μ^* denotes the exterior Lebesgue measure).