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Homework 1

– due October 29, 2012 –

1. Let μ, ν be two Radon mesures on $\mathbf{R}^n$. Consider the following two conditions:

(i) $\nu << \mu$,

(ii) $\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall A \in \mathcal{B}_n \qquad \mu(A) < \delta \implies \nu(A) < \epsilon.$

Are these conditions equivalent? If yes, provide a proof, otherwise a counterexample. What if we additionally assume that ν is finite?

2. Prove uniqueness of the Lebesgue decomposition of (nonnegative) Radon measures.

3. Let μ be a (nonnegative) Radon measure on $\mathbf{R}^n$ and let $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ be a μ-measurable function, such that $|f|^p$ is also μ-integrable, for some $p \geq 1$. Prove that:

(i) $\displaystyle \lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f \, \mathrm{d}\mu = f(x) \qquad \mu - a.a.\ x,$

(ii) $\displaystyle \lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f - f(x)|^p \, \mathrm{d}\mu = 0 \qquad \mu - a.a.\ x.$

[Hint: Use differentiation of Radon measures.]

4. Prove that:

(i)

$$\begin{aligned} \forall A \in \mathcal{M} \qquad & \mu^+(A) = \sup\{\mu(B);\ B \in \mathcal{M},\ B \subset A\}, \\ & \mu^-(A) = \sup\{-\mu(B);\ B \in \mathcal{M},\ B \subset A\}. \end{aligned}$$

(ii) There exist disjoint sets $X^+, X^- \in \mathcal{M}$ such that: $\mu^+ = \mu\lfloor X^+$ and $\mu^- = -\mu\lfloor X^-$. To do it, define X^+ to be the maximal positive set. That is $\mu(A) \geq 0$ for all $A \subset X^+$, $A \in \mathcal{M}$ and if $\tilde{X}^+$ has the same property then there must be $|\mu|(\tilde{X}^+ \setminus X^+) = 0$. Likewise, define X^- to be the maximal negative set.

5. In the context of problem 4, prove that one can have: $X = X^+ \cup X^-$ and $X^+ \cap X^- = \emptyset$. This is called the Hahn decomposition of X with respect to μ. Show that μ^+ and μ^- are mutually singular and that such decomposition $\mu = \mu^+ - \mu^-$ is unique.