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Homework 2

– due November 12, 2012 –

Let E, F, G denote Banach spaces.

1. Let $T \in \mathcal{L}(E)$ be a linear continuous operator on Banach space E. By $\mathcal{R} : \rho(T) \longrightarrow \mathcal{L}(E)$ we denote the resolvent function of T. Prove that:

(i) $\mathcal{R}(\lambda_1) - \mathcal{R}(\lambda_2) = (\lambda_1 - \lambda_2)\mathcal{R}(\lambda_1)\mathcal{R}(\lambda_1), \qquad \forall \lambda_1, \lambda_2 \in \rho(T)$

(ii) $\mathcal{R}$ is differentiable on $\rho(T)$ (find its derivative)

(iii) $\lim_{|\lambda|\to\infty} \mathcal{R}(\lambda) = 0$

(iv) the spectrum $\sigma(T)$ is always nonempty

(v) if $\lambda \in \sigma(T)$ then $\lambda^n \in \sigma(T^n)$

(vi) the spectral radius: $\max\{|\lambda|;\ \lambda \in \sigma(T)\} = \lim_{n\to\infty} \|T^n\|^{1/n}$.

2. (i) Let $T \in \mathcal{K}(E, F)$ and let $S \in \mathcal{L}(F, G)$ be injective. Prove that for every $\epsilon > 0$ there exists $C > 0$ so that:

$$\|Tx\| \leq \epsilon\|x\| + C\|STx\| \qquad \forall x \in E.$$

(ii) Let Ω be a ball in $\mathbf{R}^n$. Prove that for every $\epsilon > 0$ there exists $C > 0$ so that:

$$\|\nabla u\| \leq \epsilon\|D^2 u\| + C\|u\| \qquad \forall u \in \mathcal{C}^2(\bar{\Omega}),$$

where $\|\cdot\|$ denotes the $\mathcal{C}^0(\bar{\Omega})$ norm of a given function.

(iii) Prove that in (ii) one can take $C = 4(\min(\epsilon, 1/R))^{-1}$, where R is the radius of the ball Ω.

3. Let Ω be an open subset of $\mathbf{R}^n$. Let $K \in L^2(\Omega\times\Omega, \mathbf{K})$ and let $T \in \mathcal{L}(L^2(\Omega), L^2(\Omega))$ be the integral Hilbert-Schmidt operator:

$$(Tf)(x) = \int_\Omega K(x, y) f(y)\ \mathrm{d}y.$$

Prove that $\|T\| = \|K\|_{L^2}$ if and only if:

$$K(x, y) = K_1(x) \cdot K_2(y) \qquad \forall \text{ a.e. } x, y \in \Omega,$$

for some $K_1, K_2 \in L^2(\Omega)$.

4. Let T be the integral Hilbert-Schmidt operator, as in Problem 3. Prove that:

$$\dim \operatorname{Ker}(\mathrm{Id} - T) \leq \|K\|^2_{L^2(\Omega\times\Omega)}.$$

5. Prove that every finite codimension subspace E_0 of E is complemented in E.