Marta Lewicka, Math 2303, Fall 2012

Midterm

– December 10, 2012 –

1. (30 points) State the following results that we learned in this course. Include all details.

(i) The Fredholm alternative
(ii) The definition of the Hausdorff dimension
(iii) The Muller-Sverak theorem on convex integration.

2. (20 points) State the Lebesgue decomposition theorem for (nonnegative) Radon measures. Prove uniqueness of the stated decomposition.

Proof of uniqueness: Let μ, ν be two Radon measures on $\mathbb{R}^n$. Let $\nu = \nu_{ac}^1 + \nu_s^1 = \nu_{ac}^2 + \nu_s^2$ where the subscripts 'ac' and 's' stand for 'absolutely continuous' and 'singular'. In case $\nu_s^1, \nu_s^2 = 0$, that is when ν is absolutely continuous with respect to μ, the decomposition is unique since $\nu = \nu_{ac}$. Assume that $\nu_s^1 \neq \nu_s^2 \neq 0$. It is not possible to have two decompositions: one absolutely continuous and the other one having nontrivial singular part, since then there would be a set of μ-measure zero so that in the absolutely continuous decomposition we would obtain ν-measure zero, but the decomposition having a singular part would yield a positive value; thus the only possible case is to have two non-trivial but different singular parts. We get $\nu_s^1 - \nu_s^2 = \nu_{ac}^2 - \nu_{ac}^1$. Since we assume that ν_s^1, ν_s^2 are non-trivial and different then there is a ν-measurable set A with μ-measure zero such that (without loss of generality) $\nu_s^1(A) - \nu_s^2(A) \neq 0$. However, since A has μ-measure zero for the absolutely continuous side, this is a contradiciton.

3. (30 points) Prove that every compact self-adjoint operator T on a (complex) Hilbert space H has an eigenvalue λ satisfying $|\lambda| = \|T\|$.

Define $\lambda_1 = \sup_{\|x\|=1}\langle Tx, x\rangle$ and $\lambda_2 = \inf_{\|x\|=1}\langle Tx, x\rangle$. We know that if $\lambda_i \neq 0$ then λ_i is an eigenvalue of T. We also know that $T = 0$ iff $\langle Tx, x\rangle = 0$ for all $x \in H$. Therefore there always exists an eigenvalue λ with $|\lambda| = \sup_{\|x\|=1} |\langle Tx, x\rangle| := m$. It remains to prove that $m = \|T\|$. We have:

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=\|y\|=1} |\langle Tx, y\rangle| \geq m.$$

On the other hand, for any $\|x\| = \|y\| = 1$:

$$\begin{aligned} 2\Re\langle Tx, y\rangle = \langle Tx, y\rangle + \langle Ty, x\rangle &= \frac{1}{2}\left(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle\right) \\ &\leq \frac{1}{2}\left(\|x+y\|^2 + \|x-y\|^2\right) m = m(\|x\|^2 + \|y\|^2) = 2m \end{aligned}$$

by parallelogram identity. Therefore, $\Re\langle Tx, y\rangle \leq m$ and so, applying a rotation:

$$\sup_{\|x\|=\|y\|=1} |\langle Tx, y\rangle| \leq m.$$

Hence $\|T\| \leq m$, which proves the result.