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S. Müller · V. Šverák

Convex integration with constraints and applications to phase transitions and partial differential equations

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Abstract. We study solutions of first order partial differential relations $Du \in K$, where $u\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz map and K is a bounded set in $m \times n$ matrices, and extend Gromov's theory of convex integration in two ways. First, we allow for additional constraints on the minors of Du and second we replace Gromov's $P-$convex hull by the (functional) rank-one convex hull. The latter can be much larger than the former and this has important consequences for the existence of 'wild' solutions to elliptic systems. Our work was originally motivated by questions in the analysis of crystal microstructure and we establish the existence of a wide class of solutions to the two-well problem in the theory of martensite.

1. Introduction

We study the existence of solutions of the partial differential relation

$$Du \in K \quad \text{a.e. in } \Omega \tag{1.1}$$

subject to the boundary condition

$$u = v \quad \partial\Omega. \tag{1.2}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain, $u\colon \Omega \to \mathbb{R}^m$ is a Lipschitz map and $K \subset M^{m\times n}$ is a given subset of the $m \times n$ matrices. Such problems (and their generalizations to manifolds and jet bundles) arise in a number of areas in mathematics, Gromov's monography [Gr 86] gives an overview. Our main motivation stems from models of crystal microstructure (see [BJ 87,CK 88,BJ 92,Mu 98]). In these examples K consists of several connected components and we are therefore interested in Lipschitz

S. Müller: Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22-26, D-04103 Leipzig, Germany

V. Šverák: Department of Mathematics, University of Minnesota, 206 Church St. S.E., Minneapolis, MN 55455-0436, USA

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solutions while in many geometric applications C^1 solutions are relevant (but see [Gr 86], 2.4.11).

After the striking work by Nash [Na 54] and Kuiper [Ku 55] on the existence of nontrivial isometric C^1 immersions Gromov [Gr 73,Gr 86] developed a very general theory, called convex integration, to address (1.1) and (1.2). His main result for the Lipschitz case [Gr 86, p. 218] assures, roughly speaking, that nontrivial solutions of (1.1) and (1.2) exist if a suitable convex hull of K, called the P-convex hull, has sufficiently large interior (see [MS 96,DM 96a,DM 96b,DM 97,DM 98,Sy 98] for related work). We have recently learned that closely related ideas were already used (for the special case of elliptic systems) in Scheffer's thesis [Sch 74], see below for further discussion. For sets $K \subset M^{m\times n}$ P-convexity reduces to what was called lamination convexity in [MS 96] (Matoušek and Plecháč [MP 98] use the term set-theoretic rank-one convexity). A set $E \subset M^{m\times n}$ is lamination convex if for all matrices $A, B \in E$ that satisfy $\mathrm{rk}(B - A) = 1$ the whole segment $[A, B]$ is contained in E. The lamination convex hull E^{lc} is the smallest lamination convex set that contains E. The relevance of rank -1 convexity stems from the fact that rank -1 matrices arise exactly as gradient of maps $x \mapsto u(x \cdot n)$ that only depend on one variable. These maps are the building blocks in Gromov's construction.

In this paper we generalize Gromov's result in two directions. First we show that one can impose a constraint on a minor (subdeterminant) of Du. Such a constraint is stable under taking the lamination convex hull and thus that hull has always empty interior when all elements of K satisfy the constraint. Therefore one cannot rely on openness to construct approximate solutions but rather has to show that the constraint can be preserved at each step of the construction.

Secondly we show that the lamination convex hull can be replaced by the rank-one convex hull (called functionally rank-one convex hull in [MP 98]) which is defined by duality with rank -1 convex functions. A function $f : M^{m\times m} \to \mathbb{R}$ is rank-one convex if it is convex on every rank-one segment $[A, B]$. For a compact set K the rank -1 convex hull is defined as

$$K^{rc} = \left\{ F \in M^{m\times m} : \ f(F) \le \sup_K f, \ \forall f : M^{m\times m} \to R \ \text{ rank-one convex} \right\}.$$

For an arbitrary set U we define U^{rc} as the union of the hulls K^{rc}, for all compact sets $K \subset U$. We note in passing that in the literature the rank-one convex hull of an arbitrary set L is often defined as $(\overline{L})^{rc}$. For our purposes the separate definitions for compact and general sets are more convenient (and in line with the situation for ordinary convexity). The difference between lamination convexity (defined set-theoretically) and rank-one convexity (defined by duality with functions) may appear to be small since both notions agree for ordinary convexity but Corollary 1.5 below and the recent construction of (variational) elliptic systems with nowhere regular solutions [MS 99] show that the difference may be striking.

We now fix $t \in \mathbb{R}$ and a minor (subdeterminant) $M : M^{m\times n} \to \mathbb{R}$. We set

$$\Sigma = \{F \in M^{m\times n} : M(F) = t\}. \tag{1.3}$$

Theorem 1.1. *Let $U \subset \Sigma$ be open in Σ and bounded and let $v : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ be a piecewise linear Lipschitz map that satisfies*

$$Dv \in U^{rc} \qquad a.e. \text{ in } \Omega.$$

Assume also the the parameter t in the definition of Σ is not zero. Then there exists a Lipschitz map $u : \Omega \to \mathbb{R}^m$ that satisfies

$$\begin{aligned} Du &\in U && a.e. \text{ in } \partial\Omega \\ u &= v && \text{on } \Omega. \end{aligned}$$

The hypothesis that v be piecewise linear can be replaced by $v \in C^{2,\alpha}_{loc}(\Omega; \mathbb{R}^m)$ for some $\alpha > 0$.

The same assertion holds if Σ is replaced by $M^{m\times n}$.

Remarks. 1. For the case without constraint $C^{2,\alpha}$ can be replaced by C^1. If the constraint is on a minor of order n then $C^{2,\alpha}$ can be replaced by $C^{1,\alpha}$ (cf. Lemma 6.3 below).
2. By simple scaling and covering arguments one can see that u can be chosen so that $|u(x) - v(x)| < \varepsilon(x)$, where $\varepsilon(x)$ is a given continuous function on Ω (which can vanish at the boundary). In Gromov's terminology this means that v admits a fine approximation by solutions of $Du \in U$.

To obtain results for sets that may not be open we use Gromov's concept of an in-approximation.

Definition 1.2. *Let Σ be given by (1.3) with $t \neq 0$ and let $K \subset \Sigma$. A sequence of sets $U_i \subset \Sigma$ is an in-approximation of K in Σ, if the U_i are open in Σ and the following three conditions are satisfied:*

(i) the U_i are uniformly bounded
(ii) $U_i \subset U^{rc}_{i+1}$
(iii) $U_i \to K$ in the following sense: if $F_i \in U_i$ and $F_i \to F$ then $F \in K$.

Theorem 1.3. *Let Σ be given by (1.3) with $t \neq 0$ and let $K \subset \Sigma$. Let U_i be an in-approximation of K in Σ. Suppose that $v : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is in $C^{2,\alpha}(\Omega; \mathbb{R}^m)$ (or piecewise linear) and that*

$$Dv \in U_1 \qquad \text{in } \Omega.$$

Then there exists a Lipschitz map $u : \Omega \to R^m$ *that satisfies*

$$\begin{aligned} Du &\in K \quad a.e.\ in\ \Omega, \\ u &= v \quad on\ \partial\Omega. \end{aligned}$$

The same assertion holds if Σ *is replaced by* $M^{m\times n}$.

Remark. The condition $v \in C^{2,\alpha}_{loc}(\Omega; \mathbb{R}^m)$ can be relaxed to $v \in C^1_{loc}$ for the case without constraint and to $C^{1,\alpha}_{loc}$ if the constraint is on a minor of order n.

One application of our results concerns the so-called two-well problem in the theory of martensite (see [BJ 92,Sv 93]).

Corollary 1.4. *Suppose that* $A, B \in M^{2\times 2}$ *satisfy* $\det A = \det B = 1$ *and let* $K = SO(2)A \cup SO(2)B$. *Then the problem*

$$\begin{aligned} Du &\in K \quad a.e.\ in\ \Omega, \\ u(x) &= Fx \quad on\ \partial\Omega \end{aligned}$$

has a solution if $F \in \operatorname{int} \operatorname{conv} K$ *and* $\det F = 1$.

The next example which was found independently by several authors ([AH 86,CT 93,Sch 74,Ta 93]) illustrates the difference between lamination convexity (defined set-theoretically) and rank-one convexity (defined by duality). Let K be a subset of the diagonal 2×2 matrices given by

$$K = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \pm \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \right\} \tag{1.4}$$

(see Fig. 1). Then K contains no rank-one connections and thus $K^{lc} = K$. On the other hand K^{rc} contains the square $S = \{|F_{11}| \le 1, |F_{22}| \le 1\}$ and the segments $[A_{i+1}, J_i]$. To see this let f be a rank-one convex function that vanishes on K. Then f is convex along the horizontal and vertical lines in Fig. 1 and hence attains its maximum over S in one of the corner points of S, say at J_1. If $F(J_1) > 0$ then convexity along $[A_2, J_2]$ yields the contradiction $f(J_2) > f(J_1)$.

One can easily check that the relation $Du \in K$ only admits the trivial solution $Du = \text{const}$. Theorem 1.1 guarantees that there are maps $u : \Omega \to \mathbb{R}^2$ which vanish at $\partial\Omega$ and whose gradient remains in an arbitrarily small open neighbourhood of K.

Corollary 1.5. *Let* K *be given by (1.4) and let* $F \in K^{rc}$ *and* $\varepsilon > 0$. *Then there exists* $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ *such that*

$$\begin{aligned} \operatorname{dist}(Du, K) &< \varepsilon \quad a.e.\ in\ \Omega, \\ u(x) &= Fx \quad on\ \partial\Omega. \end{aligned}$$

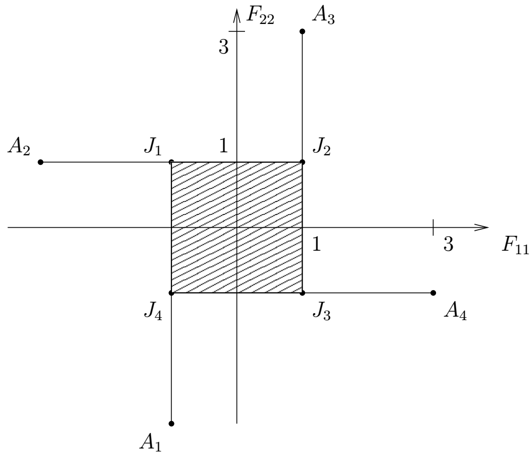

Fig. 1. The set $\{A_1, A_2, A_3, A_4\}$ is lamination convex but the rank-one convex hull contains the shaded square and the line segments $[J_i, A_{i+1}]$

The difference between rank-one convexity and lamination convexity is also relevant for the study of $m \times 2$ elliptic systems

$$\operatorname{div} Df(Dv) = 0 \tag{1.5}$$

where $v : \Omega \subset \mathbb{R}^2 \to R^m$ and $f : M^{m\times 2} \to \mathbb{R}$ is a smooth function satisfying suitable ellipticity conditions. If Ω is simply connected then (1.5) is equivalent to the partial differential relation

$$Du \in K \tag{1.6}$$

where $u : \Omega \to R^{2m}$ and

$$K = \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \in M^{2m\times 2} : Y = Df(X)J \right\}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

By a result of Ball [Ba 80] the strong ellipticity condition for (1.5) is equivalent to the condition that K contains no rank-one connection. Hence $K^{lc} = K$ for strongly elliptic f. Nonetheless there exist 2×2 systems (even in variational form) for which K^{rc} is sufficiently nontrivial, and our approach can be used to construct "wild" solutions to such systems [MS 99]. We learned recently that a closely related construction appears in the thesis of Scheffer [Sch 74]. He only discusses the nonvariational case in detail and only obtains $W^{1,2}$ solutions. Although his results are very interesting it seems that this work was never published in a journal and therefore has not received the attention it deserves.

Before discussing the main idea of the proof let us briefly mention other related work. In [MS 96] we gave a short selfcontained proof of Gromov's result for Lipschitz solutions, specialized to (1.1), (1.2) with $K \subset M^{m\times n}$. A slightly different approach based on Baire's theorem has been pursued by Dacorogna and Marcellini [DM 96a,DM 96b,DM 97], see also [Sy 98].

The case $K = O(n)$ has been studied in detail in [Gr 86], Chapter 2.4.11; for $K = O(3)$ see also [CP 95]; applications of the latter approach to other problems can also be found in [CP 97].

The proof of Theorem 1.1 relies on three steps, and a suitable approximation argument as in [MS 96] then leads to Theorem 1.3. In the first step one considers a neighbourhood U of two rank-one connected matrices A and B and shows that any affine boundary condition with gradient in $[A, B]$ can be realized. This is easy in the unconstrained case, but requires a careful approximation if a constraint on a minor is imposed. A simple induction argument yields a weaker version of Theorem 1.1 where U^{rc} is replaced by U^{lc}. In the second step we construct, under the hypothesis of Theorem 1.1, maps that satisfy $Du \in U^{rc}$ and for which set set $\{x : Du(x) \in U^{rc} \setminus U)$ has small measure. We use a result of Pedregal [Pe 93] (see also [MP 98]) that the points in K^{rc} are exactly the barycentres of certain probability measures (called laminates). We prove (and that is one of the key points) that these measures can be approximated by suitable combinations of Dirac measures that are supported in an arbitrarily small neighbourhood (in Σ) of K^{rc} (see Theorem 3.1 below). In the third step we remove the set $\{Du \in U^{rc} \setminus U\}$ by a simple iteration.

Step 1 is discussed in Sect. 2 for the unconstrained case. Step 2 is carried out in Sects. 3 and 4. In Sect. 5 we prove Theorems 1.1 and 1.3 and Corollary 1.4. Finally in Sect. 6 we carry out Step 1 for the case of a constraint.

2. The unconstrained case for a neighbourhood of two matrices

We first establish a version of Theorem 1.1 for the simplest situation, a small neighbourhood of two rank -1 connected matrices.

Lemma 2.1. *Let A and B be $m \times n$ matrices and suppose that*

$$\operatorname{rank}(B - A) = 1. \tag{2.1}$$

Let

$$C = (1-\lambda)A + \lambda B, \quad \textit{where } \lambda \in (0, 1).$$

Then for any $\delta > 0$ there exists a piecewise linear map $u : \Omega \to \mathbf{R}^m$ such that

$$\operatorname{dist}(\nabla u, \{A, B\}) \le \delta \quad \textit{a.e. in } \Omega, \tag{2.2}$$

$$\sup_{\Omega} |u(x) - Cx| \le \delta, \tag{2.3}$$

$$u(x) = Cx \quad \textit{on } \partial\Omega. \tag{2.4}$$

Proof. A simple construction was given in [MS 96]. We recall it for the convenience of the reader. We will first construct a solution for a special domain U. The argument will then be finished by an application of the Vitali covering theorem.

By an affine change of variables we may assume without loss of generality that

$$A = -\lambda a \otimes e_n, \quad B = (1-\lambda)a \otimes e_n, \quad C = 0, \quad \text{and} \quad |a| = 1.$$

Let $\varepsilon > 0$, let $V = (-1,1)^{n-1} \times ((\lambda-1)\varepsilon, \lambda\varepsilon)$ and define $v\colon V \to \mathbf{R}^m$ by

$$v(x) = -\varepsilon\lambda(1-\lambda)a + \begin{cases} -\lambda a x_n & \text{if } x_n < 0, \\ (1-\lambda)a x_n & \text{if } x_n \geq 0. \end{cases}$$

Then $\nabla v \in \{A, B\}$ and $v = 0$ at $x_n = \varepsilon(\lambda-1)$ and $x_n = \varepsilon\lambda$, but v does not vanish on the whole boundary ∂V. Next let $h(x) = \varepsilon\lambda(1-\lambda)a\sum_{i=1}^{n-1}|x_i|$. Then h is piecewise linear and $|\nabla h| = \varepsilon\lambda(1-\lambda)\sqrt{n-1}$. Set $\tilde{u} = v + h$. Note that $\tilde{u} \geq 0$ on ∂V and let $U = \{x \in V : \tilde{u}(x) < 0\}$. Then

$$\begin{gathered} \tilde{u}_{|U} \quad \text{is piecewise linear} \;, \qquad \tilde{u}_{|\partial U} = 0, \\ \operatorname{dist}(\nabla\tilde{u}, \{A, B\}) \leq \varepsilon\lambda(1-\lambda)\sqrt{n-1}, \\ |\tilde{u}| \leq \varepsilon\lambda(1-\lambda). \end{gathered}$$

By the Vitali covering theorem one can exhaust Ω by disjoint scaled copies of U. More precisely there exist $x_i \in \mathbf{R}^n$ and $r_i > 0$ such that the sets $U_i = x_i + r_i U$ are mutually disjoint and $|\Omega \setminus \cup_i U_i| = 0$. Define u by

$$u(x) = \begin{cases} r_i\tilde{u}(r_i^{-1}(x - x_i)) & \text{if} \quad x \in U_i, \\ 0 & \text{else.} \end{cases}$$

Then $\nabla u(x) = \nabla\tilde{u}(r_i^{-1}(x - x_i))$, if $x \in \Omega_i$. It follows that u is piecewise linear, that $u_{|\partial\Omega} = 0$ and that u satisfies (2.2) for a suitable choice of ε. Moreover by choosing $r_i \leq 1$ one can also satisfy (2.3). □

3. Rank-one convex functions and rank-one convex hulls

In this section we fix $m, n \in \mathbf{N}$ and we consider functions defined on (subsets of) the space $M^{m\times n}$ of all real $m \times n$ matrices. We also fix a natural number $r \leq \min(m, n)$ and a real number t. For $X \in M^{m\times n}$ we let $M(X) = \det(X_{ij})_{i,j=1}^r$ and $\Sigma = \{X \in M^{m\times n},\ M(X) = t\}$.

Let $\mathcal{O} \subset M^{m\times n}$ be an open and let $f\colon \mathcal{O} \to \mathbf{R}$ be a function. We say that f is *rank-one convex in* $\mathcal{O}$, if f is convex of each rank-one segment contained in $\mathcal{O}$. In a similar way, a function f defined on a set $\mathcal{O} \subset \Sigma$ which is open in Σ is rank-one convex in $\mathcal{O}$, if it is rank-one convex on each rank-one segment contained in $\mathcal{O}$. We will use $\mathcal{P}$ to denote the set of all compactly supported probability measures in $M^{m\times n}$. For a compact set $K \subset M^{m\times n}$ we use $\mathcal{P}(K)$ to denote the set of all probability measures supported in K. For $\nu \in \mathcal{P}$ we denote by $\bar{\nu}$ the center of mass of ν, i. e. $\bar{\nu} = \int_{M^{m\times n}} X d\nu(X)$.

A measure $\nu \in \mathcal{P}$ is a *laminate* if $\langle \nu, f\rangle \geq f(\bar{\nu})$ for each rank-one convex function $f\colon M^{m\times n} \to \mathbf{R}$. At the center of our attention will be the sets $\mathcal{M}^{\text{rc}}(K) = \{\nu \in \mathcal{P}(K),\ \ \nu \text{ is a laminate}\}$, which are defined for any compact set $K \subset M^{m\times n}$.

Let $\mathcal{O}$ be an open subset of $M^{m\times n}$or a subset of Σ which is open in Σ. We now define an important subset $\mathcal{L}(\mathcal{O})$ of laminates, called *the laminates of finite order in* $\mathcal{O}$. The definition is by induction:

1. For each $A \in \mathcal{O}$, the Dirac mass at A, denoted by δ_A, belongs to $\mathcal{L}(\mathcal{O})$.
2. Assume $\lambda_1, \ldots, \lambda_m \geq 0$ with $\sum \lambda_j = 1$, and that $\nu = \sum_{j=1}^{m} \lambda_j \delta_{A_j}$ belongs to $\mathcal{L}(\mathcal{O})$. Assume also that $[B_1, B_2]$ is a rank-one segment contained in $\mathcal{O}$, and that there is $0 \leq s \leq 1$ such that $(1-s)B_1 + sB_2 = A_m$. Then the measure $\mu = \sum_{j=1}^{m-1} \lambda_j \delta_{A_j} + (1-s)\lambda_m \delta_{B_1} + s\lambda_m \delta_{B_2}$ also belongs to $\mathcal{L}(\mathcal{O})$.

Let K be a compact subset of $M^{m\times n}$(resp. of Σ). We recall that the *rank-one convex hull* $K^{\text{rc}} \subset M^{m\times n}$ *of* K (resp. *the rank-one convex hull* $K^{\text{rc},\Sigma} \subset \Sigma$ *of* K *relative to* Σ) is defined as follows. A matrix X does not belong to K^{rc} (resp. to $K^{\text{rc},\Sigma}$) if and only if there exists $f\colon M^{m\times n} \to \mathbf{R}$ (resp. $f\colon \Sigma \to \mathbf{R}$) which is rank-one convex (resp. rank one convex in Σ) such that $f \leq 0$ on K and $f(X) > 0$. It is not difficult to see that $K^{\text{rc}} = \{\bar{\nu};\ \nu \in \mathcal{M}^{\text{rc}}(K)\}$ for any compact $K \subset M^{m\times n}$. The inclusion $\subset$ is obvious. The proof of the inclusion $\supset$ can be found in [Pe 93]; it can be also easily derived from Lemma 3.2 and Lemma 3.5 below. We can now formulate the main result of this section.

Theorem 3.1. *Let K be a compact subset of Σ and let $\nu \in \mathcal{M}^{\text{rc}}(K)$. Assume that the number t appearing in the definition of Σ is not zero. Let $\tilde{K} = K^{\text{rc},\Sigma}$ be the rank-one convex hull of K relative to Σ and let $\mathcal{O} \subset \Sigma$ be open in Σ such that $\tilde{K} \subset \mathcal{O}$. Then there exists a sequence $\nu_j \in \mathcal{L}(\mathcal{O})$ of laminates of finite order in $\mathcal{O}$ such that the ν_j converge weakly* to ν in $\mathcal{P}$.*

The statement also remains true if we replace Σ by $M^{m\times n}$ and $K^{\text{rc},\Sigma}$ by K^{rc}.

As a preparation for the proof of the theorem, we prove the following lemma.

Lemma 3.2. *Let $\mathcal{O}$ be an open subset of $M^{m\times n}$ or a subset of Σ which is open in $\mathcal{O}$. Let $f\colon \mathcal{O} \to \mathbf{R}$ be a continuous function and let $R_{\mathcal{O}} f\colon \mathcal{O} \to \mathbf{R} \cup \{-\infty\}$ be defined by $R_{\mathcal{O}} f = \sup\{g,\ g\colon \mathcal{O} \to \mathbf{R} \text{ is rank-one convex in } \mathcal{O} \text{ and } \leq f\}$. Then, for each $X \in \mathcal{O}$ we have $R_{\mathcal{O}} f(X) = \inf\{\langle \nu, f\rangle,\ \nu \in \mathcal{L}(\mathcal{O}),\ \text{and } \bar{\nu} = X\}$.*

Proof. Let us denote by $\tilde{f}$ the function in $\mathcal{O}$ defined by $\tilde{f}(X) = \inf\{\langle \nu, f\rangle, \nu \in \mathcal{L}(\mathcal{O}),$ and $\bar{\nu} = X\}$. Clearly $R_{\mathcal{O}} f \le \tilde{f}$ in $\mathcal{O}$. On the other hand, we see from the definition of the set $\mathcal{L}(\mathcal{O})$ that it has the following property: if $\nu_1, \nu_2 \in \mathcal{L}(\mathcal{O})$ and the segment $[\bar{\nu}_1, \bar{\nu}_2]$ is contained in $\mathcal{O}$, then any convex combination of ν_1 and ν_2 is again in $\mathcal{O}$. Using this, we see immediately from the definitions that $\tilde{f}$ is rank-one convex in $\mathcal{O}$ and hence $R_{\mathcal{O}} f = \tilde{f}$. □

Proof of Theorem 3.1. Let $\nu \in \mathcal{M}^{\mathrm{rc}}(K)$ and let $\bar{\nu} = A$ be its center of mass. We claim that $A \in \tilde{K}$. This can be seen as follows. First we note that $A \in \Sigma$, since $\langle \nu, M\rangle = M(\bar{\nu})$ by definition of $\mathcal{M}^{\mathrm{rc}}(K)$. If A did not belong to $\tilde{K}$, there would exist a rank-one convex function g on Σ such that $g \le 0$ on K and $g(A) > 0$. This would mean $\langle \nu, g\rangle < g(A)$, which would give a contradiction if we knew that there exists a rank-one convex function $f : M^{m\times n} \to \mathbf{R}$ such that $|f - g| \le \varepsilon$ on $K \cup \{A\}$, where ε is sufficiently small. The existence of such f is guaranteed by Lemma 3.6 below, and hence the claim $A \in \tilde{K}$ is proved. We now choose a set $U \subset \Sigma$ which is open in Σ and satisfies $\tilde{K} \subset U \subset \bar{U} \subset \mathcal{O}$. We define $\mathcal{F} = \{\mu \in \mathcal{L}(U), \ \bar{\mu} = A\}$. We claim the the weak* closure of $\mathcal{F}$ contains ν. To prove the claim, we argue by contradiction. Assume ν does not belong to the weak* closure of $\mathcal{F}$. Since $\mathcal{F}$ is clearly convex, we see from the Hahn-Banach Theorem that there exists a continuous function $f : \bar{U} \to \mathbf{R}$ such that $\langle \nu, f\rangle < \inf\{\langle \mu, f\rangle, \ \mu \in \mathcal{L}(U)$ and $\bar{\mu} = A\}$. By Lemma 3.2, we have $\inf\{\langle \mu, f\rangle, \ \mu \in \mathcal{L}(U)$ and $\bar{\mu} = A\} = R_U f(A)$. We see that the function $\tilde{f} = R_U f : U \to \mathbf{R}$ is rank-one convex in U and satisfies $\langle \nu, \tilde{f}\rangle \le \langle \nu, f\rangle < \tilde{f}(\bar{\nu})$. By Lemma 3.6 below, there exists, for each $\varepsilon > 0$, a rank-one convex function $F : M^{m\times n} \to \mathbf{R}$ such that $|F - f| \le \varepsilon$ on $\tilde{K}$. We conclude that ν cannot belong to $\mathcal{M}^{\mathrm{rc}}(K)$, a contradiction. The proof is finished.

The rest of this section is devoted to the proof of the Lemma 3.6 below. An important step in the proof of the lemma is the approximation of rank-one convex functions on Σ by smooth rank-one convex functions, a problem which we are now going to consider.

We first remark that any rank-one convex function f on $M^{m\times n}$ can be approximated by functions of the form $\varphi_\varepsilon * f$, where $\varphi_\varepsilon = \varepsilon^{-mn}\varphi(x/\varepsilon)$, with φ being a standard mollifier. If $\varphi \ge 0$, the functions $\varphi_\varepsilon * f$ are obviously rank-one convex.

To approximate rank-one convex functions on Σ by smooth functions, we will use a suitable variant of the simple mollification procedure just described. However, our method will work only for $t \ne 0$. For $t = 0$ the problem seems to be more subtle due to the singularities in Σ.

We can write each $m \times n$ matrix X as a 2×2 block matrix, $X = \begin{pmatrix} X^{11} & X^{12} \\ X^{21} & X^{22} \end{pmatrix}$, where X^{11} is an $r \times r$ matrix, X^{12} is an $(m-r) \times r$ matrix, etc. We recall that there is a natural action of the group $SL(r, \mathbf{R})$ on $M^{m \times n}$ given by $A \cdot X = \begin{pmatrix} AX^{11} & AX^{12} \\ X^{21} & X^{22} \end{pmatrix}$ (where $A \in SL(r, \mathbf{R})$ and $X \in M^{m \times n}$). This action clearly leaves Σ invariant and also maps any rank-one segment into a rank-one segment.

Let $E = \{X \in M^{m \times n},\ X^{11} = 0\}$. We consider E as an additive group which acts on $M^{m \times n}$ by $X \to X + C$, (where $X \in M^{m \times n}$ and $C \in E$). This action also preserves Σ and all rank-one segments.

We consider a family of mollifiers $\varphi_\varepsilon : SL(r, \mathbf{R}) \to \mathbf{R}$ which are smooth, non-negative, and approximate the Dirac mass at I as $\varepsilon \to 0$. Let also ψ_ε be a family of mollifiers in E which have analogous properties. For a continuous function $f \colon \Sigma \to \mathbf{R}$ we let $f_\varepsilon(X) = \int_{SL(r,\mathbf{R})} \int_E f(A \cdot (X + C))\varphi_\varepsilon(A)\psi_\varepsilon(C)\, dA\, dC$, where dA and dC denote the natural invariant measure in $SL(r, \mathbf{R})$ and E respectively. It is easy to verify that for each rank-one convex $f \colon \Sigma \to \mathbf{R}$ the functions f_ε are again rank-one convex in Σ, smooth, and converge to f uniformly on compact subsets of Σ as $\varepsilon \to 0$.

For $X \in \Sigma$ we let $n(X)$ be the unit normal to Σ satisfying $n(X) \cdot \nabla M(X) > 0$. It is well known that for $X \in M^{m \times n}$ which is sufficiently close to Σ there is a unique $\pi(X) \in \Sigma$ which is close to X such that $X = \pi(X) + tn(\pi(X))$, where $t = \operatorname{dist}(X, \Sigma)$.

Let $f \colon \Sigma \to \mathbf{R}$ be a smooth, rank-one convex function. Let U be a neighbourhood of Σ on which the projection π introduced above is well-defined. For $\varepsilon > 0$ and $k > 0$ we define $F = F_{\varepsilon,k} \colon U \to \mathbf{R}$ by $F_{\varepsilon,k}(X) = f(\pi(X)) + \varepsilon |X|^2 + k|M(X) - t|^2$, where we use the notation introduced above (see also the beginning of the section).

Lemma 3.3. *Let K be a compact subset of Σ. In the notation introduced above, for any $\varepsilon > 0$ there exists $k > 0$ such that the function $F = F_{\varepsilon,k}$ is rank-one convex in an open subset of $M^{m \times n}$ containing K.*

Proof. We argue by contradiction. Suppose the statement fails. Then there exists a sequence $A_k \in U$ converging to $A \in K$ (as $k \to \infty$) and a sequence Y_k of rank-one matrices with $|Y_k| = 1$ converging to a rank-one matrix Y with $|Y| = 1$ such that $D^2 F_{\varepsilon,k}(A_k)(Y_k, Y_k) \le 0$. Since $f \circ \pi$ is smooth in U and M is affine along all rank-one lines, we have $D^2 F_{\varepsilon,l}(A)(Y, Y) \le 0$ for each $l > 0$. Using again that $f \circ \pi$ is smooth and M is affine along all rank-one lines, we see that Y is a rank-one matrix belonging to the tangent space of Σ at A. Therefore the line described by $t \to A + tY,\ t \in \mathbf{R}$ is contained in Σ. Using the assumption that f is rank-one convex on Σ, we infer that $D^2F_{\varepsilon,l}(A)(Y,Y) \geq \varepsilon$, a contradiction. The proof of Lemma 3.3 is finished.

Lemma 3.4. *Let K be a compact subset of Σ and let $\tilde{K} = K^{\mathrm{rc},\Sigma}$ be the rank-one convex hull of K relative to Σ. Then there exists a rank-one convex $g\colon \Sigma \to \mathbf{R}$ such that $g \geq 0$ in Σ and $\tilde{K} = \{X \in \Sigma,\ g(X) = 0\}$. The statement also remains true if we replace Σ by $M^{m\times n}$ and $K^{\mathrm{rc},\Sigma}$ by K^{rc}.*

Proof. For $r > 0$ we let denote by Σ_r the set $\Sigma \cap \{|X| < r\}$. We choose $R > 0$ so that $K \subset \Sigma_{R/2}$ and define $g_1\colon \Sigma_R \to \mathbf{R}$ by

$$\begin{aligned} g_1(X) = \sup\{&f(X),\ f\colon \Sigma_R \to \mathbf{R},\\ &f \text{ is rank-one convex in } \Sigma_R \text{ and } f \leq \operatorname{dist}(\,\cdot\,, K) \text{ in } \Sigma_R\}. \end{aligned}$$

The function g_1 is obviously non-negative and rank-one convex in Σ_R. Moreover, $\{X \in \Sigma_R,\ g_1(X) = 0\} \supset K$ and from the definition of $\tilde{K}$ we see that $g_1 > 0$ outside $\tilde{K}$. We now define

$$g(X) = \begin{cases} \max\,(g_1(X), 12|X| - 9R) & \text{when } X \in \Sigma_R \\ 12|X| - 9R & \text{when } X \in \Sigma \cap \{|X| \geq R\} \end{cases}$$

Clearly g is rank-one convex on Σ in a neighbourhood of any point $X \in \Sigma$ with $|X| \neq R$. Since $g_1(X) \leq 2|X|$ when $|X| = R$, we see that we have $g(X) = 12|X| - 9$ in a neighbourhood of $\Sigma \cap \{|X| = R\}$. We infer that g is rank-one convex on Σ. The proof is easily finished.

Lemma 3.5. *Let $K \subset M^{m\times n}$ be a compact set, let $\mathcal{O}$ be an open set containing K^{rc} (the rank-one convex hull of K) and let $f\colon \mathcal{O} \to \mathbf{R}$ be rank-one convex. Then there exists $F\colon M^{m\times n} \to \mathbf{R}$ which is rank-one convex and coincides with f in a neighbourhood of K^{rc}.*

Proof. We use Lemma 3.4 to obtain a non-negative rank-one convex function $g\colon M^{m\times n} \to \mathbf{R}$ such that $K^{\mathrm{rc}} = \{X,\ g(X) = 0\}$. Replacing f by $f + c$, if necessary, we can assume that $f > 0$ in a neighbourhood of K^{rc}. For $k > 0$ we let $U_k = \{X \in \mathcal{O},\ f(X) > kg(X)\}$. We also let V_k be the union of the connected components of U_k which have a non-empty intersection with K^{rc}. It is easy to see that there exists $k_0 > 0$ such that $\bar{V}_{k_0} \subset \mathcal{O}$. We now let $F(X) = f(X)$ when $X \in V_{k_0}$ and $F(X) = k_0 g(X)$ when $X \in M^{m\times n} \setminus V_{k_0}$. It is easy to check that the function F defined in this way is rank-one convex on $M^{m\times n}$. $\square$

Lemma 3.6. *Using the notation introduced at the beginning of this section, let us assume that the number t appearing in the definition of Σ is not zero. Let $K \subset \Sigma$ be a compact set, and let $\mathcal{O} \subset \Sigma$ be a set open in Σ which contains $\tilde{K} = K^{\mathrm{rc},\Sigma}$, the rank-one convex hull of K relative* *to Σ. Let $f\colon \mathcal{O} \to \mathbf{R}$ be rank-one convex. Then, for each $\varepsilon > 0$, there exists a rank-one convex function $F\colon M^{m\times n} \to \mathbf{R}$ such that $|F - f| < \varepsilon$ on $\tilde{K}$.*

Proof. Using Lemma 3.4 we see that there exists a non-negative rank-one convex function $g\colon \Sigma \to \mathbf{R}$ such that $\tilde{K} = \{X \in \Sigma,\ g(X) = 0\}$. Let us take a large (open) ball $B \subset M^{m\times n}$ containing $\tilde{K}$. As we saw above, there exists a smooth rank-one convex function $\tilde{g}\colon \Sigma \to \mathbf{R}$ such that $|\tilde{g} - g| < \varepsilon/4$ in $\bar{B} \cap \Sigma$. By Lemma 3.3 there exists a neighbourhood U of $\Sigma \cap \bar{B}$ in $M^{m\times n}$ and a rank-one convex function $G\colon U \to \mathbf{R}$ such that $|G - g| \le \varepsilon/2$ on $\tilde{K}$. We note that the rank-one convex hull of the set $\Sigma \cap \bar{B}$ is again $\Sigma \cap \bar{B}$, and therefore we can apply Lemma 3.5. The proof is finished easily.

4. The main approximation lemma

In this section we consider a precursor to Theorem 1.1. We show that for affine boundary data $x \mapsto Fx$ with $F \in U^{rc}$ there exists a piecewise linear map u whose gradient is always in U^{rc} and most of the time in U. A simple iteration argument given in the next section will yield Theorem 1.1, and another more subtle iteration yields Theorem 1.3.

Lemma 4.1. *Let Σ be given by (1.3) with $t \neq 0$. Let V be an open set in Σ, let $F \in V^{rc}$ and let $\varepsilon > 0$. Then there exists a piecewise linear map $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ such that $Du \in V^{rc}$ a.e. in Ω and*

$$|\{Du \notin V\}| < \varepsilon\,|\Omega|,$$
$$u(x) = Fx \qquad \text{on } \partial\Omega.$$

The same assertion holds if Σ is replaced by $M^{m\times n}$.

Proof. By definition there exists a compact set $K \subset V$ such that $F \in K^{rc}$. In view of [Pe 93] (see also Sect. 3) there exists a probability measure $\nu \in \mathcal{M}^{rc}(K)$ such that $F = \overline{\nu} = \langle \nu, id\rangle$. Using the action of the group $SL(r, \mathbf{R}) \times E$ on Σ defined in Sect. 3, we see that V^{rc} is open in Σ. Theorem 3.1 yields the existence of laminates of finite order $\nu_j \in \mathcal{L}(V^{rc})$ that converge to ν in the weak* topology and satisfy $\overline{\nu_j} = F$.

It only remains to show that for each $\mu \in \mathcal{L}(V^{rc})$, $\mu = \sum_{i=1}^{l} \lambda_i \delta_{A_i}$ and each $\varepsilon > 0$ there exists a piecewise linear map $u\colon \Omega \to \mathbb{R}^m$ that satisfies $Du \in V^{rc}$ and

$$\Big|\big|\{|Du - A_i| < \varepsilon\}\big| - \mu(A_i)|\Omega|\Big| < \varepsilon\,|\Omega|, \tag{4.1}$$
$$u = \overline{\mu}\,x \qquad \text{on } \partial\Omega. \tag{4.2}$$

We prove this assertion by induction over the order of the laminate. For laminates $\mu = \lambda\delta_A + (1-\lambda)\delta_B$ of order one the assertion follows from Theorem 6.1 (and the fact that $\int_\Omega Du = \overline{\mu}\,|\Omega|$). Assume that the assertion holds for laminates of order k (or less) and let $\mu \in \mathcal{L}(V^{rc})$ be a laminate of order $k+1$. Then there exists a laminate μ' of order k, $\mu' = \sum_{i=1}^{l-1} \lambda_i' \delta_{A_i'}$ and matrices $A_{l-1}, A_l \in V^{rc}$ with $\mathrm{rk}(A_l - A_{l-1}) = 1$ and $A_{l-1}' = sA_{l-1} + (1-s)A_l$, $s \in (0,1)$ such that

$$\mu = \mu' - \lambda_{l-1}'\delta_{A_{l-1}'} + s\lambda_{l-1}'\delta_{A_{l-1}} + (1-s)\lambda_{l-1}'\delta_{A_l}.$$

By the induction assumption there exists, for each $\delta > 0$, a piecewise linear map $v : \Omega \to \mathbb{R}^m$ such that $Dv \in V^{rc}$

$$\left| \left| \{|Dv - A_i'| < \delta\} \right| - \mu'(A_i')|\Omega| \right| < \delta|\Omega|,$$

$$v = \overline{\mu'}\,x = \overline{\mu}\,x \qquad \text{on } \partial\Omega.$$

Consider the set $E \subset \Omega$ where $|Dv - A_{l-1}'| < \delta$. Then E is a countable union of open sets on which v is affine (up to a set of measure two) and $|\,|E| - \mu(A_{l-1}')|\Omega|\,| < \delta|\Omega|$. Hence we may choose a subset E' such that $|\,|E'| - \lambda_{l-1}|\Omega|\,| < 2\delta|\Omega|$ and E' is a finite union of open sets E_j on which v is affine.

It remains to modify v on these sets. Let $F_j = Du_{|E_j}$. Then $|F_j - A_{l-1}'| < \delta$, and we claim that there exist $B_j, C_j \in V^{rc}$ such that $F_j = sB_j + (1-s)C_j$, $\mathrm{rk}(B_j - C_j) = 1$ and $|B_j - A_{l-1}| < C\delta$, $|C_j - A_l| < C\delta$, where C may depend on A_{l-1} but not on δ. Indeed in the case without constraint one can take $B_j = A_{l-1} + (F_j - A_{l-1}')$, $C_j = A_l + (F_j - A_{l-1}')$ and the assertion $B_j, C_j \in V^{rc}$ follows for a sufficiently small choice of δ. If a constraint is imposed one can use the group action on Σ as in Sect. 3 instead of the translation on $M^{m\times n}$ to define B_j and C_j.

Using Theorem 6.1 we can replace v on each E_j by a map u which satisfies $u = v$ on ∂E_j, $Du \in V^{rc}$ and $|\,|\{|Du - B_j| < \delta\}| - s|E_j|\,| < \delta|E_j|$, $|\,\{|Du - C_j| < \delta\}| - (1-s)|E_j| < \delta|E_j|$. If δ is chosen sufficiently small (in dependence on ε, μ and V^{rc}) then u satisfies (4.1) and (4.2). This finishes the proof of the lemma for the case with constraint. The unconstrained case is completely analogous and was treated in [MS 99]. □

5. Proof of the main results

Theorem 1.1 is obtained by an iteration of Lemma 4.1 which removes the set where $Du \notin V$. Theorem 1.3 can be deduced from Theorem 1.1 by a careful choice of approximations $u^{(i)}$ with $Du^{(i)} \in U_i$. The argument is the same as in [MS 96]. Since it is short, we repeat it for the convenience of the reader.

Proof of Theorem 1.1. We only consider the situation with constraint since the unconstrained case is analogous. Suppose first that the boundary data v are affine, $v(x) = Fx + a$. Let $\varepsilon > 0$. By Lemma 4.1 there exists a piecewise linear map $u^{(1)}\colon \Omega \to \mathbb{R}^m$ that satisfies $u^{(1)} = v$ on $\partial\Omega$ and

$$Du^{(1)} \in U^{rc}, \quad |\{Du^{(1)} \notin U\}| < \varepsilon|\Omega|.$$

Since $u^{(1)}$ is piecewise linear there exists a family of disjoint sets Ω_k such that $|\Omega \setminus \bigcup \Omega_k| = 0$ and $u^{(1)}_{|\Omega_k}$ is affine. Let $\{\Omega^{(1)}_j\}$ be the subfamily of those sets where $Du^{(1)} \notin U$. Applying Lemma 4.1 to each set $\Omega^{(1)}_j$ we find maps $u^{(2)}_j$ that satisfy $u^{(2)}_j = u^{(1)}$ on $\partial\Omega^{(1)}_j$ and

$$Du^{(2)}_j \in U^{rc}, \quad |\{Du^{(2)}_j \notin U\}| < \varepsilon|\Omega^{(1)}_j|.$$

Let

$$u^{(2)} = \begin{cases} u^{(1)} & \text{on } \Omega \setminus \bigcup \Omega^{(1)}_j, \\ u^{(2)}_j & \text{on } \Omega^{(1)}_j. \end{cases}$$

Then $u^{(2)} = v$ on $\partial\Omega$, $Du^{(2)} \in U^{rc}$ a.e. and

$$|\{Du^{(2)} \notin U\}| < \varepsilon^2|\Omega|,$$
$$|\{Du^{(2)} \neq Du^{(1)}\}| < \varepsilon|\Omega|.$$

Repeating this process we find maps $u^{(k)}$ such that $u^{(k)} = v$ in $\partial\Omega$, $Du^{(k)} \in U^{rc}$ a.e.

$$|\{Du^{(k)} \notin U\}| < \varepsilon^k|\Omega|,$$
$$|\{Du^{(k)} \neq Du^{k-1)}\}| < \varepsilon^{k-1}|\Omega|.$$

In particular $Du^{(k)} \to Du$ in measure and $Du \in U$ a.e. This finishes the proof for affine v.

If v is piecewise affine it suffices to apply the previous argument to each region where v is affine. Finally if $v \in C^{2,\alpha}_{loc} \cap W^{1,\infty}$ then we can first approximate v by a piecewise affine map (see Lemma 6.3 and the remarks following it). □

Proof of Theorem 1.3. Again it suffices to consider affine boundary data $v = Fx$. For piecewise linear data one can argue on each region where v is affine, and for general data one can use Lemma 6.3 to obtain a piecewise linear approximation. Let $F \in U^{(1)}$, $\delta > 0$. By Theorem 1.1 there exists a piecewise linear map $u^{(2)}$ such that $Du^{(2)} \in U^{(2)}$ a.e. in Ω, $u^{(2)} = v$ on $\partial\Omega$ and

$$\|u^{(2)} - v\|_\infty < \delta.$$

Given $u^{(i)}$ (with $Du^{(i)} \in U^{(i)}$) and $\delta_i > 0$ we obtain ε_i, δ_{i+1} and $u^{(i+1)}$ inductively as follows. Let

$$\Omega_i = \{x \in \Omega\colon\ \operatorname{dist}(x, \partial\Omega) > 2^{-i}\},$$

let $\varrho \in C_0^\infty(\mathbb{R}^n)$ with $\int \varrho = 1$ be a usual mollifier and let $\varrho_\varepsilon(x) = \varepsilon^{-n}\varrho(x/\varepsilon)$. Then there exist $\varepsilon_i < 2^{-i}$ such that

$$\|\varrho_{\varepsilon_i} * Du^{(i)} - Du^{(i)}\|_{\Omega_i} < 2^{-i}.$$

Let

$$\delta_{i+1} = \min(2^{-i}\varepsilon_i, \delta_i\varepsilon_i/2).$$

Then there exists $u^{(i+1)}$ with $Du^{(i+1)} \in U^{(i+1)}$ a.e. and

$$\|u^{(i+1)} - u^{(i)}\|_\infty < \delta_{i+1}.$$

Since $\Sigma\delta_i < \infty$ we have $u^{(i)} \to \overline{u}$ uniformly and $u^{(i)} = v$ on $\partial\Omega$. Moreover

$$\begin{aligned} R_i = \left\|\varrho_{\varepsilon_i} * (Du^{(i)} - D\overline{u})\right\|_{L^1(\Omega_i)} &\le \left\|D\varrho_{\varepsilon_i} * (u^{(i)} - \overline{u})\right\|_{L^1(\Omega_i)} \\ &\le \frac{c}{\varepsilon_i}\sum_{j=i+1}^\infty \delta_j \le \frac{2c}{\varepsilon_i}\delta_i \quad \to 0,\ \text{as } i \to \infty, \end{aligned}$$

and thus

$$\begin{aligned} \left\|Du^{(i)} - D\overline{u}\right\|_{L^1(\Omega)} \le \left\|Du^{(i)} - D\overline{u}\right\|_{L^1(\Omega\setminus\Omega_i)} + \left\|\varrho_{\varepsilon_i} * Du^{(i)} - Du^{(i)}\right\|_{L^1(\Omega_i)} \\ + \left\|\varrho_{\varepsilon_i} * D\overline{u} - D\overline{u}\right\|_{L^1(\Omega_i)} + R_i \quad \to 0,\ \text{as } i \to \infty, \end{aligned}$$

since $|Du^{(i)}| + |D\overline{u}| \le C$. Hence $Du^{(i)} \to D\overline{u}$ in L^1 and thus $D\overline{u} \in K$. □

Proof of Corollary 1.4. Let

$$\Sigma = \left\{F \in M^{2\times 2}\colon\ \det F = 1\right\}.$$

Then [Sv 93]

$$K^{rc} = K^{rc,\Sigma} = \Sigma \cap \operatorname{conv} K$$

and the sets

$$\begin{aligned} U_1 &= \Sigma \cap \operatorname{int}\operatorname{conv} K \\ U_i &= \left\{F \in U_1\colon 0 < \operatorname{dist}(F, K) < 2^{-i}\right\} \end{aligned}$$

are an in-approximation of K in Σ.

6. The basic construction with constraints

6.1. Main result

Theorem 6.1. *Let $A, B \in M^{m\times n}$, $n, m \geq 2$ and let M be a minor of order $r \geq 2$. Suppose that*

$$\operatorname{rank}(B - A) = 1, \quad M(A) = M(B) \neq 0.$$

Let

$$C = (1 - \lambda)A + \lambda B, \text{ where } \lambda \in (0, 1).$$

Then for any $\delta > 0$, there exists a piecewise linear map $u : \Omega \to \mathbb{R}^n$ such that

$$M(Du) = M(A) \qquad a.e. \text{ in } \Omega, \tag{6.1}$$

$$\operatorname{dist}(Du, [A, B]) \leq \delta \qquad a.e. \text{ in } \Omega, \tag{6.2}$$

$$|\{\operatorname{dist}(Du, \{A, B\}) > \delta\}| \leq \delta, \tag{6.3}$$

$$\sup |u - Cx| \leq \delta, \tag{6.4}$$

$$u = Cx \qquad on\ \partial\Omega. \tag{6.5}$$

Remarks. 1. If $m = n$ is even one can construct symplectic maps rather than volume preserving maps in a similar way. Also certain linear constraints such $\operatorname{div} u = 0$ or $Du = (Du)^T$ can be handled (cf. the construction of ψ and v in the proof of Lemma 6.2 below).

2. The proof employs approximation arguments that are simple but lead to a construction that is hard to visualize. For $n = m = 2$ a direct construction involving 20 gradients is possible. It even satisfies $\operatorname{dist}(Du, \{A, B\}) \leq \delta$ a. e. in Ω.

To prove Theorem 6.1 we first construct smooth functions that satisfy (6.1)–(6.5) and then employ a general argument to approximate those by piecewise linear functions. Moreover we may assume $M(A) = M(B) = 1$. Note that if suffices to establish (6.1)–(6.3) and (6.5) for $\Omega = (0, 1)^n$. The result for general Ω and (6.4) can then be deduced by covering and scaling as in the proof of Lemma 2.1.

6.2. Approximation by C^∞ maps

Lemma 6.2. *Let A, B, C as in Theorem 6.1 and suppose that $\Omega = (0, 1)^n$, $r \geq 2$ and*

$$M(A) = M(B) = 1.$$

Then there exists $u \in C^\infty(\bar{\Omega})$ that satisfies (6.1)–(6.3) and (6.5).

Proof. We begin with the typical case

$$r = m = n, \quad M(Du) = \det Du.$$

After a linear change of variables we may assume

$$A = Id - \lambda\, e_1 \otimes \nu, \quad B = Id + (1-\lambda)\, e_1 \otimes \nu, \quad C = Id.$$

Since $\det A = \det C$ we must have $\nu_1 = \nu \cdot e_1 = 0$ and we may assume $\nu_2 \neq 0$.

The map u is obtained as the flow of a divergence free vector field. Let $h : \mathbb{R} \to \mathbb{R}$ be a smooth one-periodic function that satisfies $h'' \in [-\lambda, 1-\lambda]$, $|\{h'' \notin \{-\lambda, 1-\lambda\}\} \cap [0,1]| < \delta/4$ and let $\eta \in C_0^\infty(U)$, $U = (0,1)^n$, be a suitable cut-off function. Define the vector field $v \in C_0^\infty(U; \mathbb{R}^r)$ by

$$\begin{aligned}
\psi(x) &= \frac{\varepsilon^2}{\nu_2}\eta(x)h\left(\frac{x \cdot \nu}{\varepsilon}\right)\\
v^1 &= \partial_2\psi = \varepsilon\eta(x)h'\left(\frac{x \cdot \nu}{\varepsilon}\right) + \frac{\varepsilon^2}{\nu_2}(\partial_2\eta)h\left(\frac{x \cdot \nu}{\varepsilon}\right)\\
v^2 &= -\partial_1\psi = -\frac{\varepsilon^2}{\nu_2}(\partial_1\eta)(x)h\left(\frac{x \cdot \nu}{\varepsilon}\right)\\
v^3 &= \cdots = v^r = 0.
\end{aligned}$$

The small parameter $\varepsilon > 0$ will be chosen below. Consider the flow φ_t generated by v:

$$\frac{d}{dt}\varphi_t(x) = v(\varphi_t(x)), \quad \varphi_0 = id.$$

We claim that $u = \varphi_1$ satisfies (6.1)–(6.3), (6.5). Indeed (6.1) and (6.5) hold since v is divergence free and has compact support.

To prove the remaining assertions let $F_t = D\varphi_t$. Then

$$\frac{d}{dt}F_t = (Dv \circ \varphi_t)F_t, \quad F_0 = id. \tag{6.6}$$

Now

$$Dv(x) = \eta(x)h''\left(\frac{x \cdot \nu}{\varepsilon}\right)(e_1 \otimes \nu) + O(\varepsilon)$$

and, for $t \in [0,1]$,

$$|(\varphi_t(x) - x) \cdot \nu| \le t \sup |v \cdot \nu| \le \sup |v^2| \le C\varepsilon^2,$$

since $\nu_1 = 0$. Thus (6.6) yields

$$Du(x) = F_1(x) = e^{L(x)} + O(\varepsilon),$$

where

$$L(x) = \eta(x)h''\left(\frac{x \cdot \nu}{\varepsilon}\right)(e_1 \otimes \nu)\,.$$

Estimates (6.2) and (6.3) now follow from the properties of h after a suitable choice of η and ε, since

$$e^{t(a \otimes b)} = Id + t\, a \otimes b \;\text{ if }\; a \cdot b = 0.$$

This finishes the proof in the typical case $r = m = n$.

Now consider the case

$$r = m < n.$$

We may assume that the minor M involves the first m rows and columns. For $x \in \mathbb{R}^n$ we let $x = (x', \tilde{x})$, where $x' = (x_1, \dots, x_m)$, $\tilde{x} = (x_{m+1}, \dots, x_n)$ and similarly we write $F = (F'|\tilde{F})$ for an $m \times n$ matrix F. We may assume

$$C = (Id\,|\,0),\;\; A = C - \lambda\, e_1 \otimes \nu,\;\; B = C + (1-\lambda)\, e_1 \otimes \nu.$$

Suppose first $\nu' \neq 0$. Then we may suppose $\nu_2 \neq 0$ and define the vector field v as before. Consider the flow φ_t given by

$$\frac{d}{dt}\varphi_t(x) = v(\varphi_t(x), \tilde{x}), \quad \varphi_0(x) = x'.$$

We claim that $u = \varphi_1$ has the desired properties. Indeed if we consider $F_t = D\varphi_t$ and $\Phi_t(x) = (\varphi_t(x), \tilde{x})$ we have

$$\begin{aligned} \frac{d}{dt}F_t' &= (Dv)' \circ \Phi_t \;\; F_t' & F_0' &= Id,\\ \frac{d}{dt}\tilde{F}_t &= (Dv)' \circ \Phi_t \;\; \tilde{F}_t + \tilde{D}v & \tilde{F}_0 &= 0. \end{aligned}$$

In particular we have $M(F_t) = \det F_t' = 1$ since $\operatorname{tr}(Dv)' = 0$. Moreover $|\Phi_t(x) - x) \cdot \nu| \leq C\varepsilon^2$ and the other estimates follows as for the case $r = n = m$.

If $\nu' = 0$ one can still use the same construction provided that ψ is redefined as follows

$$\psi(x) = \varepsilon^3 \eta(x) h\left(\frac{x_2}{\varepsilon} + \frac{x \cdot \nu}{\varepsilon^2}\right).$$

Finally consider the case

$$r < m.$$

We may suppose that

$$C = \left(\begin{array}{c|c} Id & 0 \\ \hline 0 & 0 \end{array}\right), \quad a = \begin{pmatrix} \alpha e_1 \\ \hat{a} \end{pmatrix} \text{ and } \nu_1 = 0 \text{ if } \alpha \neq 0.$$

Let $v^1, \ldots, v^r$ be the vector field that is appropriate for the situation obtained by deleting the rows $r+1, \ldots, m$ and define maps $\varphi_t : (0,1)^n \to \mathbb{R}^m$ by

$$\begin{aligned} \frac{d}{dt}\varphi_t^i(x) &= \alpha\, v^i\big(\varphi_t^1(x),\ \ldots, \varphi_t^r(x), x^{r+1}, \ldots, x^n\big), \\ \varphi_0^i(x) &= x^i, \qquad\qquad \text{for } i = 1, \ldots, r, \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}\varphi_t^i(x) &= \hat{a}^i v^1\left(\varphi_t^1(x), \ldots, \varphi_t^r(x), x^{r+1}, \ldots, x^n\right), \\ \varphi_0^i(x) &= 0, \qquad\qquad \text{for } i = r+1, \ldots, m. \end{aligned}$$

To see that the last equation yields the desired result one uses the fact that $|D\varphi_t^i - e_i|$ is small for $i = 2, \ldots, r$ and that $\nu_1 = 0$ if $\alpha \neq 0$. □

6.3. Approximation by piecewise linear maps

To finish the proof of Theorem 6.1, we note that if we prove the result up to condition (6.4), then (6.4) can be achieved by a suitable scaling and the use of Vitali's covering theorem. Therefore it only remains to establish the following approximation result.

Lemma 6.3. *Let Ω be a bounded open set in $\mathbb{R}^n$ and let M be a minor of order $r \geq 2$ and let $\alpha > 0$. Suppose that $u \in C^{2,\alpha}_{loc}(\Omega, \mathbb{R}^m) \cap W^{1,\infty}(\Omega; \mathbb{R}^m)$ and*

$$M(Du) = 1 \quad in\ \Omega.$$

Then for every $\delta > 0$ there exists a piecewise linear map $v : \Omega \to \mathbb{R}^m$ that satisfies

$$M(Dv) = 1 \qquad in\ \Omega, \tag{6.7}$$

$$\|Du - Dv\|_{L^\infty} \leq \delta, \tag{6.8}$$

$$u = v \qquad on\ \partial\Omega. \tag{6.9}$$

Remarks. 1. If $r = n$, then $C^{2,\alpha}_{loc}$ can be replaced by $C^{1,\alpha}_{loc}$.
2. If U is an open subset of $\Sigma = \{F \in M^{m\times n} : M(F) = 1\}$ and $Du \in U$ then one can achieve in addition $Dv \in U$.

Consider first the typical case $m = n = r$, $M(Du) = \det Du$. The main idea is that on a ball $B(a, r)$ where $r^\alpha [Du]_\alpha$ is sufficiently small one can replace u by a map with the same boundary values that is affine on $B(a, r/2)$. To achieve the replacement one can first consider an interpolation between $u(a) + Du(a)x$ and u in $B(a, r) \setminus B(a, r/2)$ and then use the following result of Dacorogna and Moser to reestablish the constraint.

Lemma 6.4 ([DM 90]). *Let U be a smooth and bounded domain in $\mathbb{R}^n$. For $k \geq 1$ and $\alpha \in (0, 1)$ consider the set*

$$X = \left\{ u \in C^{k,\alpha}(U; \mathbb{R}^n) : \int_U \det Du \, dx = |U| \right\}.$$

There exists a neighbourhood $\mathcal{U}$ of the identity map in X and a smooth map $\mathcal{L} : \mathcal{U} \to C^{k,\alpha}(U; \mathbb{R}^n)$ such that for all $u \in \mathcal{U}$ the map $\mathcal{L}u$ is a diffeomorphism and

$$\begin{aligned} \det D\,\mathcal{L}u &= 1 \qquad \text{in } U, \\ \mathcal{L}u &= u \qquad \text{on } \partial U. \end{aligned}$$

Moreover $\mathcal{L}\,id = id$.

Proof. Choosing the neighbourhood $\mathcal{U}$ sufficiently small we may assume that it consists of diffeomorphisms. By Lemma 4 of [DM 90] there exists an operator Φ from a neighbourhood $\mathcal{V}$ of the constant 1 in $Y = \{f \in C^{k-1,\alpha}(\Omega) : \int_\Omega f = |\Omega|\}$ to a neighbourhood of the identity in X such that

$$\begin{aligned} \det D\Phi(f) &= f \qquad \text{in } U, \\ \Phi(f) &= id \qquad \text{on } \partial U. \end{aligned}$$

It easily follows from the construction of Φ via the contraction principle that Φ is actually a smooth map (estimate (4) on p. 11 of [DM 90] is incorrect, but their results are correct; for the present purpose it suffices that the estimate in question holds with $\|w_i\|_0$ replaced by $\|w_i\|_{k,\alpha}$). Now define $\mathcal{L}$ by

$$\mathcal{L}u = \Phi(f) \circ u, \qquad f = \det Du^{-1},$$

Then $\mathcal{L}u$ satisfies (6.10) and (6.11). Since multiplication and composition are smooth operations in $C^{k,\alpha}(\Omega; \mathbb{R}^n)$ and the map $u \mapsto u^{-1}$ is smooth in $\mathcal{U}$, the map $u \to \det Du^{-1}$ is a smooth map from $\mathcal{U}$ to $\mathcal{V}$ (if $\mathcal{U}$ is sufficiently small), and $\mathcal{L}$ is smooth. Finally $\mathcal{L}\,id = id$ since $\Phi(1) = id$. □

Corollary 6.5. *For each $n \in \mathbb{N}, \alpha > 0, M > 0$ there exist a $\delta > 0$ such that the following holds. If $B(a, r) \subset \mathbb{R}^n$, $u \in C^{1,\alpha}(B(a, r); \mathbb{R}^n)$ and*

$$\begin{aligned} \det Du &= 1 & &\text{in } \Omega, \\ r^\alpha [Du]_\alpha &\leq \delta, & &\|Du\|_{L^\infty} \leq M, \end{aligned}$$

then there exists a $\tilde{u} \in C^{1,\alpha}_{loc}(B(a, r) \setminus \partial B(a, r/2))$ such that

$$\begin{aligned} \det D\tilde{u} &= 1, \\ D\tilde{u} &= Du(0) \qquad \text{in } B(a, r/2), \\ \tilde{u} &= u \qquad \text{on } \partial B(a, r), \end{aligned}$$

$$r^{-1}\|u - \tilde{u}\|_{L^\infty} + \|Du - D\tilde{u}\|_{L^\infty} \leq C(M)\delta.$$

Proof. By scaling and translation we may suppose $a = 0, r = 1, u(a) = 0$. First suppose that $Du(0) = Id$. For $\varphi \in C_0^\infty(B(0, 1)), \varphi \equiv 1$ on $B(0, 1/2))$ consider the interpolation

$$\hat{u}(x) = \varphi(x)x + (1 - \varphi(x))u(x)$$

on $B_1 \setminus B_{1/2}$. If $\delta < \bar{\delta}(n, \alpha)$, then we can apply Lemma 6.4 and define

$$\tilde{u} = \begin{cases} x & \text{on } B_{1/2}, \\ \mathcal{L}\hat{u} & \text{on } B_1 \setminus B_{1/2}. \end{cases}$$

If $F := Du(0) \neq Id$ then we can first consider $v = F^{-1}u$, define $\tilde{v}$ as before and let $\tilde{u} = F\tilde{v}$. Since $\|F^{-1}\| = \|\operatorname{adj} F\| \leq CM^{n-1}$ the assertion follows for $\delta < C^{-1}M^{1-n}\bar{\delta}(n, \alpha)$. □

Proof of Lemma 6.3. It suffices to show the following assertion:

There exists a constant $\Theta > 0$ (which only depends on n, m and r) with the following property. For each $\delta > 0$, each $\alpha > 0$ and each pair (u, Ω) that satisfies the hypotheses of the lemma there exists a map $\tilde{u} \in W^{1,\infty}(\Omega, \mathbb{R}^n)$, a finite number of disjoint closed sets $A_j \subset \Omega$, a closed null set N and a $\beta > 0$ such that

$$\begin{aligned} M(D\tilde{u}) &= 1 && \text{in } \Omega, \\ \tilde{u}_{|A_j} & && \text{is affine}, \\ \left|\bigcup A_j\right| &\geq \Theta|\Omega|, \\ \|\tilde{u} - u\|_{W^{1,\infty}} &\leq \delta, \\ \tilde{u} &\in C^{1,\beta}_{loc}\big(\Omega \setminus (N \cup \bigcup A_j)\big), \\ \tilde{u} - u &\in W_0^{1,\infty}(\Omega; \mathbb{R}^n). \end{aligned}$$

Indeed if the assertion holds, then one can inductively obtain a decreasing sequence of open sets Ω_i and maps $u^{(i)}$ such that $u^{(0)} = u$, $u^{(i+1)} - u^{(i)} \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)$,

$$\begin{aligned} \|u^{(i+1)} - u^{(i)}\|_{W^{1,\infty}} &\leq 2^{-i-1}\delta, \\ u^{(i+1)} &= u^{(i)} && \text{on } \Omega \setminus \Omega_i, \\ |\Omega_{i+1}| &\leq (1 - \Theta)|\Omega_i|, \end{aligned}$$

and $\Omega \setminus \Omega_i$ is a finite union of closed sets (up to a closed null set N_i) on each of which $u^{(i)}$ is affine. It follows that $u^{(i)} \to v$ in $W_0^{1,\infty}$, $u - v \in W_0^{1,\infty}$ and v is piecewise linear. Moreover $\|v - u\|_{W^{1,\infty}} \leq \delta$.

To prove the assertion we first consider the typical case $r = n = m$,

$$M(Du) = \det Du.$$

There exists open sets $\Omega'' \subset\subset \Omega' \subset\subset \Omega$ (where $\subset\subset$ denotes compact inclusion, i.e. $\overline{\Omega'} \subset \Omega$ etc.) with $|\Omega''| \geq \frac{1}{2}|\Omega|$. If $\varrho > 0$ is sufficiently small then Ω'' can be covered by a lattice of disjoint open cubes of size ϱ that are contained in Ω'. The $C^{1,\alpha}$ norm of u is uniformly bounded on the cubes. If we choose ϱ sufficiently small then each cube contains a ball of radius r to which Corollary 6.5 applies. This yields the assertion for $r = n = m$.

The same reasoning applies for general $r \geq 2$ if we replace Corollary 6.5 by Lemma 6.6 below. We only state it for $r = m$, since for $r < m$ one can simply use usually interpolation by cut-off functions for the components $u^{r+1}, \ldots, u^m$. □

To fix the notation we write $x = (x', y) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$, we denote the derivative with respect to the first m components by D' and the derivatives with respect to the remaining components by D_y. Finally we sometimes write $m \times n$ matrices as $F = (F', G) \in M^{m\times n} \times M^{(n-m)\times m}$.

Lemma 6.6. *Let $V = B_2 \times B_2 \subset \mathbb{R}^m \times \mathbb{R}^{n-m}$ and $\alpha > 0$. Then there exist $\delta > 0$, $M > 0$ such that for all $u \in C^{2,\alpha}(V; \mathbb{R}^m)$ the following holds. If*

$$\|Du - Du(0)\|_{1,\alpha} < \delta, \quad \sup |D'u| \leq M, \quad \det D'u = 1,$$

then there exists a Lipschitz map $v : V \to \mathbb{R}^m$ that satisfies

$$\begin{aligned} \det D'v &= 1 && \text{in } V,\\ Dv &= Du(0) && \text{in } B_{1/2} \times B_{1/2},\\ v &= u && \text{in } \Omega \setminus B_1 \times B_1,\\ \sup |Dv - Du| &< C\delta, && \\ v &\in C^{2,\alpha/2}(V \setminus N), && \end{aligned}$$

where $N = (\partial B_{1/2} \cup \partial B_1) \times \overline{B_1}$.

The proof of Lemma 6.6 relies on Lemma 6.4 and the following characterization of the nonuniqueness in Lemma 6.4.

Lemma 6.7. *Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary. For $k \geq 2$ and $\alpha \in (0, 1)$ consider the spaces*

$$\begin{aligned} X_k &:= C^{k,\alpha}_{\partial,tr} := \left\{ v \in C^{k,\alpha}(\overline{\Omega}; \mathbb{R}^n) : \ \operatorname{div} v = 0 \ \text{ in } \Omega, v = 0 \ \text{ on } \partial\Omega \right\},\\ Y_k &:= D^{k,\alpha}_{\alpha,\det} := \{ \varphi : C^{k,\alpha}(\overline{\Omega}; \mathbb{R}^n) : \ \varphi : \ \overline{\Omega} \to \overline{\Omega} \ \text{diffeomorphism},\\ &\qquad \det D\varphi = 1 \ \text{ in } \Omega, \varphi = id \ \text{ on } \partial\Omega \}. \end{aligned}$$

Then there exists a smooth diffeomorphism exp *that maps a small neighbourhood of* 0 *in X_k onto a neighbourhood of the identity map in Y_k.*

Proof. The Lemma follows from general results on the geodesic flow in the group of volume preserving diffeomorphisms (see [Ar 66,EM 70], [HKMRS 98]). Since we need to use the H^1 metric (rather than the standard L^2 metric) to enforce the boundary condition $\varphi = id$ on $\partial\Omega$ (rather than $\varphi(\partial\Omega) = \partial\Omega$) we sketch the proof in the appendix.

Proof of Lemma 6.6. We first construct a map w that has the right properties in the inner cylinder $B_{1/2} \times B_2$. An important point is that $\int_{B_{1/2}} \det D'w\,dx' = |B_{1/2}|$. We then define an extension $u_0 = \psi(x')w + (1-\psi(x'))u$ that agrees with u outside $B_1 \times B_1$. In general $\det D'u_0 \neq 1$. By Lemma 6.4 we can replace $u_0(\cdot, y)$ by a map $\tilde{u}(\cdot, y)$ that satisfies $\det D'u = 1$ and agrees with $u_0(\cdot, y)$ on $\partial(B_1 \setminus B_{1/2})$. Finally we modify $\tilde{u}$ for $|y| \geq 3/4$ with the help of the exponential map defined in Lemma 6.7 so that it agrees with u on $B_1 \setminus B_{1/2} \times \partial B_1$.

We may suppose that

$$F := \; Du(0) = (Id, 0)$$

since otherwise we could consider

$$\hat{u}(x', y) = u((F')^{-1}x', y) - Gy,$$

where $F = (F', G)$. In addition we may assume $u(0) = 0$.

Step 1: Construction of w.
Let $\varphi \in C_0^\infty(B_{3/4})$, $\varphi_{|B_{1/2}} \equiv 1$, and define

$$\begin{aligned}
\tilde{w}(x', y) &= \varphi(y)x' + (1-\varphi(y))u(x', y),\\
\lambda(y) &= \frac{1}{|B_{1/2}|}\int_{B_{1/2}} \det D'\tilde{w}(x', y)\,dx'\\
w(x', y) &= \lambda^{-1/m}(y)\tilde{w}(x', y).
\end{aligned}$$

Then w is as smooth as u and

$$\int_{B_{1/2}} \det D'w\,dx' = |B_{1/2}|,$$

$$\begin{aligned}
w(x', y) &= x' &&\text{on } B_2 \times B_{1/2},\\
w(x', y) &= u(x', y) &&\text{on } B_2 \times B_2 \setminus B_{3/4}.
\end{aligned}$$

Note that $\lambda \in C^{2,\alpha}$ although it may appear at first glance that one derivative is lost in the definition of λ. Indeed, using the formula $\operatorname{div}' \operatorname{cof} D'\tilde{w} = 0$ we obtain

$$\begin{aligned}
|B_{1/2}|\partial_{y_i}\lambda &= \int_{B_{1/2}} (\operatorname{cof} D'\tilde{w} : D'\partial_{y_i}\tilde{w})\,dx'\\
&= \int_{\partial B_{1/2}} (\operatorname{cof} D'\tilde{w} : \partial_{y_i}\tilde{w} \otimes \nu')\,dx',
\end{aligned}$$

where ν' is the outer normal of $\partial B_{1/2}$. This yields the desired regularity of λ. Further inspection shows that $\|D\tilde{w} - (id, 0)\|_{1,\alpha} \leq C\,\delta$ and $\|Dw - (id, 0)\|_{1,\alpha} \leq C\,\delta$. In particular all the maps $w(\cdot, y)$ are diffeomorphisms of B_2 if $\delta > 0$ is sufficiently small.

Step 2: Interpolation between w and u.
Let $\psi \in C_0^\infty(B_1)$, $\psi_{|B_{1/2}} \equiv 1$ and let

$$u_0(x', y) = \psi(x')w(x', y) + (1 - \psi(x'))u(x', y).$$

Then $Du_0(0) = (id\,|\,0)$ and $\|Du_0 - (id\,|\,0)\|_{1,\alpha} \leq C\,\delta$. In particular all the maps $u_0(\cdot, y)$ are diffeomorphisms of B_2. Moreover

$$u_0 = \begin{cases} w & \text{on } \partial B_{1/2} \times B_2, \\ u & \text{on } \partial B_1 \times B_2, \end{cases}$$

and

$$\begin{aligned} \int_{B_1 \setminus B_{1/2}} \det D'u_0\, dx' &= \int_{B_1} \det D'u_0\, dx' - \int_{B_{1/2}} \det D'u_0\, dx' \\ &= \int_{B_1} \det D'u\, dx' - \int_{B_{1/2}} \det D'w\, dx' \\ &= |B_1 \setminus B_{1/2}|. \end{aligned}$$

Step 3: Projection of u_0 onto volume preserving maps.
Let

$$\Omega = B_1 \setminus \overline{B_{1/2}}.$$

For fixed y we can apply Lemma 6.4 with U replaced by Ω (and n replaced by m) to $u_0(\cdot, y)$. Let

$$\tilde{u}(\cdot, y) = \mathcal{L}u_0(\cdot, y).$$

Then for $k = 1, 2$

$$\begin{aligned} \|\tilde{u}(\cdot, y)\|_{k,\alpha,\Omega} &\leq C\,\|u_0(\cdot, y)\|_{k,\alpha,\Omega} \\ &\leq \|u\|_{k,\alpha,V}. \end{aligned}$$

Moreover by Lipschitz continuity of $\mathcal{L}$

$$\begin{aligned} \|\tilde{u}(\cdot, y) - \tilde{u}(\cdot, \hat{y})\|_{k,\alpha/2,\Omega} &\leq C\ \|u_0(\cdot, y) - u_0(\cdot, \hat{y})\|_{k,\alpha,\Omega} \\ &\leq C\,\|u\|_{k,\alpha,V}\ |y - \hat{y}|^{\alpha/2}. \end{aligned}$$

Using the differentiability of $\mathcal{L}$ one finds similarly

$$\|D_y^l\tilde{u}(\cdot, y) - D_y^l\tilde{u}(\cdot, \hat{y})\|_{k-l,\alpha/2,\Omega} \leq C\ \|u\|_{k,\alpha,V}|y - \hat{y}|^{\alpha/2}.$$

Hence $\tilde{u} \in C^{2,\alpha/2}(\Omega' \times B_2)$.

Step 4: Modification of $\tilde{u}$ for $|y| > 3/4$.
For $|y| > 3/4$ define

$$\eta(\cdot, y) = (u_0(\cdot, y))^{-1} \circ \tilde{u}(\cdot, y) = (u(\cdot, y))^{-1} \circ \tilde{u}(\cdot, y).$$

Then η measures the 'difference' between $\tilde{u}$ and u and

$$\begin{aligned} \eta(\cdot, y) &= id \quad \text{on } \partial\Omega = \partial B_1 \cup \partial B_{1/2} \\ \det D'\eta &= 1. \end{aligned}$$

Now let $\psi \in C_0^\infty(B_1)$, $\psi = 1$ on $B_{7/8}$ and let

$$\begin{aligned} v(\cdot, y) &= \tilde{u}(\cdot, y) && \text{on } \Omega \times B_{3/4} \\ v(\cdot, y) &= u(\cdot, y) \circ \exp(\psi(y) \exp^{-1} \eta(\cdot, y)) && \text{on } \Omega \times B_1 \setminus B_{3/4} \\ v(\cdot, y) &= w(\cdot, y) && \text{on } B_{1/2} \times B_1 \\ v &= u && \text{elsewhere.} \end{aligned}$$

Then v has the desired properties. □

References

[AH 86] R. Aumann, S. Hart: Bi-convexity and bi-martingales. Israel J. Math. **54**, 159–180 (1986)

[Ar 66] V.I. Arnold: Sur la geometrié differentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluids parfaits. Ann. Inst. Grenoble **16**, 319–361 (1966)

[AK 98] V.I. Arnold, B.A. Khesin: Topological methods in hydrodynamics. Springer, 1998

[Ba 80] J.M. Ball: Strict convexity, strong ellipticity and regularity in the calculus of variations. Math. Proc. Cambridge Phil. Soc. **87**, 501–513 (1980)

[BJ 87] J.M. Ball, R.D. James: Fine phase mixtures as minimizers of energy. Arch. Rat. Mech. Anal. **100**, 13–52 (1987)

[BJ 92] J.M. Ball, R.D. James: Proposed experimental tests of a theory of fine microstructure and the two-well problem. Phil. Trans. Roy. Soc. London A **338**, 389–450 (1992)

[CT 93] E. Casadio-Tarabusi: An algebraic characterization of quasi-convex functions. Ricerche Mat. **42**, 11–24 (1993)

[CP 97] P. Celada, S. Perrotta: Functions with prescribed singular values of the gradient. preprint SISSA 56/97/M, 1997

[CP 95] A. Cellina, S. Perrotta: On a problem of potential wells. J. Convex Anal. **2**, 103–115 (1995)

[CK 88] M. Chipot, D. Kinderlehrer: Equilibrium configurations of crystals. Arch. Rat. Mech. Anal. **103**, 237–277 (1988)

[DM 96a] B. Dacorogna, P. Marcellini: Théorèmes d'existence dans les cas scalaire et vectoriel pour les équations de Hamilton-Jacobi. C.R.A.S. Paris, Serie I **322**, 237–240 (1996)

[DM 96b] B. Dacorogna, P. Marcellini: Sur le problème de Cauchy-Dirichlet pour les systèmes d'équations non linéaires du premier ordre. C.R.A.S. Paris, Serie I **323**, 599–602 (1996)

[DM 97] B. Dacorogna, P. Marcellini: General existence theorems for Hamilton-Jacobi equations in the scalar and vectorial cases. Acta Math. **178**, 1–37 (1997)

[DM 98] B. Dacorogna, P. Marcellini: Cauchy-Dirichlet problem for first order nonlinear systems. J. Funct. Anal. **152**, 404–446 (1998)

[DM 90] B. Dacorogna, J. Moser: On a partial differential equation involving the Jacobian determinant. Ann. IHP Anal. non linéaire **7**, 1–26 (1990)

[EM 70] D.G. Ebin, J.Marsden: Groups of diffeomorphisms and the motion of an incompressible fluid. Ann. Math. **92**, 102–163 (1970)

[GR 86] V. Girault, P.-A. Raviart: Finite element methods for Navier-Stokes equations. Springer, 1986

[Gr 73] M. Gromov: Convex integration of differential relations. Izw. Akad. Nauk S.S.S.R. **37**, 329–343 (1973)

[Gr 86] M. Gromov: Partial differential relations. Springer, 1986

[HKMRS 98] D.D. Holm, S. Kouranbaeva, J.E. Marsden, T. Ratiu, S. Skholler: A nonlinear analysis of the averaged Euler equations, Preprint

[Ku 55] N.H. Kuiper: On C^1 isometric embeddings, I., Nederl. Akad. Wetensch. Proc. A **58**, 545–556 (1955)

[MP 98] J. Matoušek, P. Plecháč: On functional separately convex hulls. J. Discrete Comput. Geom. **19**, 105–130 (1998)

[Mu 98] S. Müller: Variational models for microstructure and phase transitions. to appear in: Proc. C.I.M.E summer school 'Calculus of variations and geometric evolution problems' (S. Hildebrandt, M. Struwe, eds.), http://www.mis.mpg.de/cgi-bin/lecturenotes.pl

[MS 96] S. Müller, V. Šverák: Attainment results for the two-well problem by convex integration. Geometric analysis and the calculus of variations (J. Jost, ed.), International Press, 239–251 (1996)

[MS 99] S. Müller, V. Šverák: Convex integration of Lipschitz functions and counterexamples to regularity. Preprint

[Na 54] J. Nash: C^1 isometric embeddings. Ann. Math. **60**, 383–396 (1954)

[Pe 93] P. Pedregal: Laminates and microstructure. Europ. J. Appl. Math. **4**, 121–149 (1993)

[Sch 74] V. Scheffer: Regularity and irregularity of solutions to nonlinear second order elliptic systems of partial differential equations and inequalities. Dissertation, Princeton University, 1974

[Sv 93] V. Šverák: On the problem of two wells. In: Microstructure and phase transitions, IMA Vol. Appl. Math. **54** (D. Kinderlehrer, R.D. James, M. Luskin, J. Ericksen, eds.) Springer, 1993, pp. 183–189

[Sy 98] M.A. Sychev: Comparing various methods of resolving nonconvex variational problems. Preprint SISSA 66/98

[Ta 93] L. Tartar: Some remarks on separately convex functions. in: Microstructure and phase transitions, IMA Vol. Math. Appl. **54** (D. Kinderlehrer, R.D. James, M. Luskin, J.L. Ericksen, eds.) Springer, 1993, pp. 191–204

Appendix: Proof of Lemma 6.7

Proof. The result is well-known to experts and a detailed proof in a more general situation is given in [HKMRS 98]. We sketch a proof for the case at hand for the convenience of the reader. The map exp will be constructed as the time-one map of the flow generated by a suitable (time-dependent) vector field v. If we were interested in volume-preserving diffeomorphisms that satisfy merely $\varphi(\partial\Omega) = \partial\Omega$ we could take v as the solution of the incompressible Euler equations. To achieve $\varphi_{|\partial\Omega} = id$ we use a variant of the Euler equations where the orthogonal projection onto divergence free vector fields is taken with respect to the H_0^1 scalar product rather than the L^2 scalar product.

Arnold [Ar 66,AK 98] observed that the flow generated by the solutions of the Euler equations corresponds to geodesics in the group of volume preserving diffeomorphisms, equipped with a translation invariant metric given by the L^2 scalar product on vector fields, see [EM 70] for a detailed analysis. The flow discussed below corresponds to geodesics with respect to a metric induced by an H_0^1 scalar product. This motivated the notation exp for the map. We are grateful to J.E. Marsden for pointing out to us that the resulting equations are known as the averaged Euler equations and that a detailed study will appear in a series of papers beginning with [HKMRS 98].

To fix the notation let $H = H_0^1(\Omega; \mathbb{R}^n)$, with scalar product $(u, v) = \int_\Omega Du\, Dv$, and let $Y = \{v \in H : \operatorname{div} v = 0\}$ denote the closed subspace of divergence free vector fields. The orthogonal complement Y is given by (see e.g. [GR 86], Chapter I, Cor. 2.3)

$$Y^\perp = \{\Delta_D^{-1} \nabla p : p \in L^2(\Omega)\}$$

where $\Delta_D = H_0^1 \to H^{-1}$ is the Dirichlet Laplacian. For $u \in H$ the orthogonal projection Qu onto $Y^\perp$ is given by

$$Qu = \Delta_D^{-1} \nabla p,$$

where p is the unique solution of

$$\operatorname{div} \Delta_D^{-1} \nabla p = \operatorname{div} u.$$

If we let

$$T = \operatorname{div} \Delta_D^{-1} \nabla$$

then T is an isomorphism of $L^2_{av} = \{p \in L^2(\Omega) : \int_\Omega p = 0\}$ onto itself (see [GR 86], Chapter I, Cor. 2.3 and 2.4). Moreover by standard regularity arguments one sees that T is also an isomorphism of $C^{k,\alpha}_{av} = \{p \in C^{k,\alpha}(\overline{\Omega}) : \int_\Omega p = 0\}$ and Q is a bounded operator from $C^{k,\alpha}_\partial = \{v \in C^{k,\alpha}(\overline{\Omega}; \mathbb{R}^n) : v = 0 \text{ on } \partial\Omega\}$ onto the subspace $C^{k,\alpha}_{\partial,div}$ of divergence free vector fields.

To motivate the definition of the flow consider a one-parameter family of diffeomorphisms $\eta_t : \Omega \to \Omega$, $t \in I$ and let $v = \left(\frac{d}{dt}\eta_t\right) \circ \eta_t^{-1}$. For any function $f : I \times \Omega \to \mathbb{R}$ define the material time derivative by

$$\dot{f} = \partial_t f + (v \cdot D) f = \left[\frac{d}{dt}(f \circ \eta_t)\right] \circ \eta_t^{-1}.$$

In particular

$$\dot{v} = \left(\frac{d^2}{dt^2}\eta_t\right) \circ \eta_t^{-1}. \tag{A.1}$$

The simplest evolution equation would be $\dot{v} = 0$ but this does not preserve the constraint of being divergence free. To keep the constraint a natural approach is to look for an equation which yields

$$\dot{v} \in Y^{\perp}.$$

Now

$$\begin{aligned} Q\dot{v} &= Q\,\partial_t v + Q(v \cdot D)v \\ &= \partial_t\, Qv + (v \cdot D)Qv - [v \cdot D, Q]v \\ &= (Qv)^{\cdot} - [v \cdot D, Q]v. \end{aligned} \tag{A.2}$$

Since we expect v to remain divergence free (i.e. $Qv = 0$) this suggest to study the equation

$$\dot{v} = -[v \cdot D, Q]v. \tag{A.3}$$

The key point is now that the commutator $[v \cdot D, Q]$ does not lose derivatives (although each term in the commutator does).

Claim: The map

$$(u, v) \mapsto C_u v := [u \cdot D, Q]v$$

is a bounded bilinear (and hence smooth) map from $C_\partial^{k,\alpha} \times C_\partial^{k,\alpha} \to C_\partial^{k,\alpha}$ for $k \geq 2$.

Now (A.1) in conjunction with (A.3) yields the following initial-value problem for η_t

$$\begin{aligned} &\frac{d^2}{dt^2}\eta_t = \mathcal{N}\left(\eta_t, \frac{d}{dt}\eta_t\right), \\ &\frac{d}{dt}\eta_t = v_0,\ \eta_0 = id, \end{aligned} \tag{A.4}$$

where

$$\mathcal{N}(q, w) = \left(C_{w \circ q^{-1}} w \circ q^{-1}\right) \circ q.$$

Since composition yields a smooth map $C_\partial^{k,\alpha} \times D_\partial^{k,\alpha} \to C_\partial^{k,\alpha}$ for $k \geq 1$ the claim implies that $\mathcal{N}$ is smooth and by the general theory of odes in Banach manifolds the initial value problem (A.4) has a solution for $t \in (-2, 2)$ if $\|v_0\|_{k,\alpha}$ is sufficiently small. Moreover the map $\exp : v_0 \mapsto \eta_1$ is smooth in a neighbourhood of 0 and $D\exp_{|0} = id$. It suffices thus to verify that the η_t are volume preserving or, equivalently, that v remains divergence free, provided that $v_0 \in Y_k$ (see Lemma 6.7 for the notation). Lemma 6.7 then follows from the implicit function theorem.

To see that v remains divergence free rewrite (A.3) as

$$\partial_t v = -(v\cdot D)v - (v\cdot D)Qv + Q[(v\cdot D)v].$$

Since $Q^2 = Q$ this yields

$$\partial_t\, Qv = Q\partial_t v = -Q[(v\cdot D)Qv]$$

and

$$\begin{aligned}(Qv)^{\cdot} &= \partial_t\, Qv + (v\cdot D)Q\, Qv\\ &= [v\cdot D, Q]\, Qv.\end{aligned}$$

Since $Qv_0 = 0$, boundedness of the commutator and Gronwall's inequality imply that $Qv \equiv 0$, hence $\operatorname{div} v = 0$. To see this in detail one can define the material description $(Qv)_m := Qv\circ\eta_t$. Then $(Qv)^{\cdot} = \left(\frac{d}{dt}(Qv)_m\right)\circ\eta_t^{-1}$ and hence

$$\begin{aligned}\frac{d}{dt}(Qv)_m &= \left([v\cdot D, Q]\,(Qv)_m\circ\eta_t^{-1}\right)\circ\eta_t\\ \frac{d}{dt}\left\|(Qv)_m\right\|_{k,\alpha} &\le \left\|\frac{d}{dt}(Qv)_m\right\|_{k,\alpha}\\ &\le C\left\|(Qv)_m\right\|_{k,\alpha}\end{aligned}$$

and $(Qv)_m(0) = Qv_0 = 0$.

It only remains to prove the claim that the bilinear map $(u, v) \mapsto [u\cdot D, Q]v$ is bounded from $C^{k,\alpha}_\partial \times C^{k,\alpha}_\partial$ to $C^{k,\alpha}_\partial$. Note that Q can be written as

$$Q = \Delta_D^{-1}\nabla T^{-1}\operatorname{div}, \quad T = \operatorname{div}\Delta_D^{-1}\nabla.$$

The assertion now follows from repeated use of the (formal) commutator relations

$$\begin{aligned}[][A, B_1B_2] &= [A, B_1]B_2 + B_1[A, B_2],\\ B[A, B^{-1}] &= -[A, B]B^{-1}\end{aligned}$$

and

$$\begin{aligned}[][u\cdot D, D]f &= u_j\partial_j\nabla f - \nabla(u_j\partial_j f)\\ &= -(\nabla u_j)\partial_j f,\\ [u\cdot D, \operatorname{div}] &= u_j\partial_j(\partial_i v_i) - \partial_i(u_j\partial_j\partial_i v_i)\\ &= -(\partial_i u_j)(\partial_j v_i).\end{aligned}$$

We leave it to the courageous reader to verify that for $u, v \in C_0^\infty$ all formal operations are justified. Note that

$$\int_\Omega (u \cdot D) f = -\int_\Omega (\operatorname{div} u) f$$

and thus application of $(u \cdot D)$ does not preserve C_{av}^∞ so that $[A, T^{-1}]$ is not defined. If π denotes the projection $f \to f - \frac{1}{|\Omega|}\int_\Omega f$, then Q can be written as $Q = \Delta_D^{-1} \nabla \pi T^{-1} \pi \operatorname{div}$, and the commutator $\pi[A, T^{-1}\pi]$ is defined and has the desired properties. $\square$