Quasiconvexity and Partial Regularity in the Calculus of Variations

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Abstract

We prove partial regularity of minimizers of certain functionals in the calculus of variations, under the principal assumption that the integrands be uniformly strictly quasiconvex. This is of interest since quasiconvexity is known in many circumstances to be necessary and sufficient for the weak sequential lower semicontinuity of these functionals on appropriate Sobolev spaces. Examples covered by the regularity theory include functionals with integrands which are convex in the determinants of various submatrices of the gradient matrix.

1. Introduction

This paper establishes the partial regularity of minimizers for certain functionals in the calculus of variations, described as follows. Let n, N be positive integers, denote by $M^{n\times N}$ the space of all real $n\times N$ matrices, and suppose $\Omega \subset \mathbb{R}^n$ is open, bounded, and smooth. Then, for $v: \Omega \to \mathbb{R}^N$, consider the functional

$$(1.1) \qquad I[v] \equiv \int_\Omega F(Dv)\, dy,$$

where

$$Dv \equiv \left(\left(\frac{\partial v^i}{\partial x_\alpha}\right)\right) \quad (1 \leqq \alpha \leqq n, l \leqq i \leqq N)$$

is the gradient matrix of v and

$$F: M^{n\times N} \to \mathbb{R}$$

is given. Two principal tasks of the calculus of variations are (a) to prove the existence of minimizers of $I[\cdot]$ subject to given, but here unspecified, boundary conditions and (b) to study the smoothness of such minimizers.

For n, $N > 1$ the real breakthrough regarding problem (a) was MORREY's paper [14], which isolated a property of F which is necessary and sufficient, in many circumstances, for the weak sequential lower semicontinuity of $I[\cdot]$ on certain Sobolev spaces. This condition is that F be *quasiconvex*,* namely

$$\int_O F(A)\,dy \leqq \int_O F(A + D\phi)\,dy \tag{1.2}$$

for all smooth, bounded, open domains $O \subset \mathbb{R}^n$, all matrices $A \in M^{n\times N}$, and all $\phi \in C^1(O; \mathbb{R}^N)$ with $\phi = 0$ on ∂O. MORREY's techniques have recently been refined by ACERBI & FUSCO, who proved the following.

Theorem 0 ([1]). *Assume* $F: M^{n\times N} \to \mathbb{R}$ *is continuous and*

$$0 \leqq F(P) \leqq C(1 + |P|^q) \quad (P \in M^{n\times N}) \tag{1.3}$$

for some constant C *and* $1 \leqq q < \infty$. *Then* $I[\cdot]$ *is weakly sequentially lower semicontinuous on the Sobolev space* $W^{1,q}(\Omega, \mathbb{R}^N)$ *if and only if* F *is quasiconvex.*

When $1 < q < \infty$, Theorem 0 and an additional coercivity assumption of the form

$$F(P) \geqq b\,|P|^q, \quad (b > 0,\ P \in M^{n\times N}) \tag{1.4}$$

imply the existence of at least one minimizer of $I[\cdot]$ for given Dirichlet boundary conditions (cf. MORREY [15, Thm. 4.4.7]). This result and its various extensions provide a fairly satisfactory understanding of problem (a). Additionally, BALL [4], [5] has modified MORREY's ideas and derived important existence assertions for several problems in nonlinear elasticity. A related paper concerning quasiconvexity and uniqueness is KNOPS & STUART [12]; see also MEYERS [13] for higher order problems.

Progress concerning the regularity problem (b) has to date been less definitive. When $q = 2$, and F is a uniformly strictly convex C^2 function with bounded second derivatives, GIUSTI & MIRANDA [11], using a blow-up argument, and GIUSTI & GIAQUINTA [9], using direct methods, have shown that any minimizer u has Hölder continuous first derivatives on some open set $\Omega_0 \subset \Omega$, with $H^{n-p}(\Omega \setminus \Omega_0) = 0$ for some $p > 2$ (cf. also MORREY [16]). This in turn implies that u is C^∞ on Ω_0 provided F is C^∞, though examples (cf. [8, p. 51–63]) demonstrate that the singular set may be nonempty. Furthermore, these regularity proofs require that F be convex, a hypothesis which is strictly stronger than quasiconvexity and so not satisfied by many interesting examples to which the existence theorems apply.

This paper partly closes the gap between the existence and regularity theorems by demonstrating the partial regularity of minimizers of $I[\cdot]$ under the principal assumption that F be *uniformly strictly quasiconvex*, that is,

$$\int_O (F(A) + \gamma\,|D\phi|^2)\,dy \leqq \int_O F(A + D\phi)\,dy \tag{1.5}$$

* The reader should be warned that this is MORREY's original terminology, which he changed in writing section 4.4 of his book [15].

for some $\gamma > 0$ and all smooth, bounded open domains $O \subset \mathbb{R}^n$, all matrices $A \in M^{n\times N}$, and all $\phi \in C^1(O; \mathbb{R}^N)$ with $\phi = 0$ on ∂O. Stated succinctly, the advance here is to take quasiconvexity, the primary hypothesis of the lower semicontinuity theorems, and to show that, slightly strengthened, it forces partial regularity. The basic new technical observation is that (1.5) permits the proof of a so-called "CACCIOPPOLI inequality" for a minimizer, after which the partial regularity follows easily from a variant of the blow-up method of GIUSTI & MIRANDA [11].

Full statements of the hypotheses and theorems may be found in section 2 below. As the proof of the main Theorem 2 is rather complicated, I have also chosen to present a simpler case, Theorem 1, whose proof displays the main ideas; this is given in sections 3–4. Sections 5–7 contain the proof of Theorem 2, while the concluding section 8 describes a broad class of interesting examples covered by Theorem 2 but beyond the scope of previous theory. These include functionals with integrands which are convex in the determinants of various submatrices of Dv.

Remarks on connections with geometric measure theory. After this work was substantially completed I looked at the survey paper [2] of ALLARD & ALMGREN and realized the strong similarities between the results described above and various lower semicontinuity and partial regularity theorems in geometric measure theory, especially in the work of ALMGREN, FEDERER and BOMBIERI. Indeed, if S is an n-rectifiable subset of $\mathbb{R}^{n+N}$ we may define

$$I[S] \equiv \int_S F(\Sigma)\, dH^n,$$

where $\Sigma = \Sigma(y)$ denotes the approximate tangent n-plane to S at (H^n) almost all $y \in S$,

$$F: G(N+n, n) \to \mathbb{R}$$

is given, and $G(N+n, n)$ denotes the Grassmann manifold of n-plane directions in $\mathbb{R}^{n+N}$. By definition, $I[\cdot]$ is *elliptic* provided there exists $\gamma > 0$ such that

$$\gamma[H^n(S) - H^n(D)] \leqq I[S] - I[D] \tag{1.6}$$

for each flat n-disk D in $\mathbb{R}^{n+N}$ and each n-rectifiable S with $\partial S = \partial D$. Then basic theorems of geometric measure theory assert (i) that $I(\cdot)$ and its various generalizations to more complicated n dimensional "surfaces" are lower semicontinuous with respect to appropriate topologies (cf. FEDERER [7, 5.1.5]) and (ii) that minimizers of $I[\cdot]$ are in fact C^1 manifolds except possibly for a "small" singular set (cf. ALMGREN [3], ALLARD & ALMGREN [3, 7–10], and FEDERER [7, 5.3.14–5.3.17]). These regularity proofs likewise use blow-up arguments (introduced originally by DE GIORGI, whose work was historically the inspiration for GIUSTI & MIRANDA [11]).

I am not sufficiently versed in geometric measure theory to understand fully the connections, but it is clear that the regularity theorems in this paper are analogues for nonparametric calculus of variations problems of the results for parametric problems discovered by ALMGREN [3]. Note in particular that since

$\phi = 0$ on ∂O we can rewrite (1.5) in the form

$$\gamma \int_O (|A + D\phi|^2 - |A|^2)\, dy \leqq \int_O (F(A + D\phi) - F(A))\, dy, \tag{1.7}$$

and this is certainly analogous to the ellipticity condition (1.6). More to the point, a recent paper of BOMBIERI [6], which gives a new proof of partial regularity for minimizing and almost minimizing currents, seems to have some technical points in common with this paper; compare, in particular, Lemmas 10 and 11 in [6] with the Caccioppoli inequalities, Lemmas 3.1 and 5.1, derived below. All this being said, I should also note that the calculus of variations—partial differential equations conclusions and the geometric measure theoretic results remain dissimilar in several important ways: for example, in the latter theory there are difficulties in showing that the minimizer is locally a graph, whereas in the former theory there are problems concerning the growth of F and its derivatives at infinity.

In any case, it is certainly clear that the accomplishments of geometric measure theory outshadow the current state of non-parametric calculus of variations and nonlinear elliptic systems. I hope that this paper, although none of its results will surprise geometric measure theory experts, will nonetheless prove interesting to others. In particular, whereas the analogy between uniform strict quasiconvexity (1.7) and ellipticity (1.6) is obvious, I have not been able to find any explicit mention of this in print.

Notation. For the most part I have adopted the notion of GIAQUINTA [8]. In particular I use the summation convention that Latin indices run from 1 to N and Greek indices from 1 to n, while the letter C throughout denotes various constants depending only on known quantities.

Other notational conventions are these:

$$B(x, r) \equiv \{y \in \mathbb{R}^n : |x - y| < r\}, \quad B(r) \equiv B(0, r), \quad B \equiv B(1).$$

$$|A|^2 \equiv a^i_\alpha a^i_\alpha, \quad (A = ((a^i_\alpha)) \in M^{n\times N})$$

$$\fint_\Omega f\, dx = \frac{1}{|\Omega|} \int_\Omega f\, dx,$$

$$(u)_{x,r} \equiv \fint_{B(x,r)} u\, dy, \quad (Du)_{x,r} \equiv \fint_{B(x,r)} Du\, dy.$$

$$(u)_r = (u)_{0,r}, \quad (Du)_r = (Du)_{0,r}.$$

$$C^1_0(\Omega) \equiv \{\phi \in C^1(\bar{\Omega}) : \phi = 0 \quad \text{on } \partial\Omega\}.$$

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2. Hypotheses and statements of theorems

Throughout we assume that the function $F: M^{n\times N} \to \mathbb{R}$ is at least twice continuously differentiable.

As noted in the introduction it seems best first to present the proof for a special case. Accordingly let us now suppose $\gamma > 0$ and that

$$\text{(H1)} \qquad \int_{B(x,r)} (F(A) + \gamma\,|D\phi|^2)\,dy \leqq \int_{B(x,r)} F(A + D\phi)\,dy$$

for all $x \in \mathbb{R}^n$, $r > 0$, $A \in M^{n\times N}$ and $\phi \in C_0^1(B(x,r);\mathbb{R}^N)$. Assume also that

$$\text{(H2)} \qquad |D^2F(P)| \leqq C$$

for some constant C and all $P \in M^{n\times N}$.

Hypothesis (H2) implies that

$$\text{(2.1)} \qquad |F(P)| \leqq C(1 + |P|^2), \quad |DF(P)| \leqq C(1 + |P|) \quad (P \in M^{n\times N})$$

for some appropriate constant C, and thus (H1) is valid also for $\phi \in H_0^1(B(x,r);\mathbb{R}^N)$.

We now call $u \in H^1(\Omega;\mathbb{R}^N)$ a *minimizer* of $I[\cdot]$ provided

$$\text{(2.2)} \qquad I[u] \leqq I[u + \phi]$$

for every $\phi \in H_0^1(\Omega;\mathbb{R}^N)$.

Theorem 1. *Suppose F satisfies* (H1), (H2) *and let $u \in W^{1,\infty}(\Omega;\mathbb{R}^N)$ be a minimizer of $I[\cdot]$. Then there exists an open subset Ω_0 of Ω such that*

$$|\Omega \setminus \Omega_0| = 0$$

and

$$Du \in C^\alpha(\Omega_0; M^{n\times N})$$

for each $0 < \alpha < 1$.

Remark. If, in addition, $F \in C^\infty(M^{n\times N};\mathbb{R})$, a standard bootstrap argument shows that $u \in C^\infty(\Omega_0;\mathbb{R}^N)$.

The proof of Theorem 1 is fairly straightforward, but unfortunately the hypotheses that u be Lipschitz and D^2F be bounded are too restrictive and exclude many interesting examples (see section 8). We therefore next modify our approach to allow for polynomial growth of D^2F; this in turn requires a modification of the quasiconvexity hypothesis. Thus assume that q, $2 \leqq q < \infty$, is given and that F satisfies, for some $\gamma > 0$,

$$\text{(H3)} \qquad \int_{B(x,r)} (F(A) + \gamma(1 + |D\phi|^{q-2})\,|D\phi|^2)\,dy \leqq \int_{B(x,r)} F(A + D\phi)\,dy$$

for all $x \in \mathbb{R}^n$, $r > 0$, $A \in M^{n\times N}$, and $\phi \in C_0^1(B(x,r);\mathbb{R}^N)$. Suppose also that

$$\text{(H4)} \qquad |D^2F(P)| \leqq C(1 + |P|^{q-2})$$

for some constant C and all $P \in M^{n\times N}$.

Assumption (H4) implies

(2.3) $\quad |F(P)| \leqq C(1+|P|^q), \quad |DF(P)| \leqq C(1+|P|^{q-1}) \quad (P \in M^{n\times N})$

for some appropriate constant C, and so (H3) is valid also for $\phi \in W_0^{1,q}(B(x_0, r); \mathbb{R}^N)$.

We call $u \in W^{1,q}(\Omega; \mathbb{R}^N)$ a *minimizer* of $I[\cdot]$ provided

(2.4) $$I[u] \leqq I[u+\phi]$$

for every $\phi \in W_0^{1,q}(\Omega; \mathbb{R}^N)$.

Theorem 2. *Assume that* $2 \leqq q < \infty$, *the function* F *satisfies* (H3), (H4), *and* $u \in W^{1,q}(\Omega; \mathbb{R}^N)$ *is a minimizer of* $I[\cdot]$. *Then there exists an open subset* Ω_0 *of* Ω *such that*

$$|\Omega \setminus \Omega_0| = 0$$

and

$$Du \in C^\alpha(\Omega_0; M^{n\times N})$$

for each $0 < \alpha < 1$.

Remark. As before if also $F \in C^\infty(M^{n\times N}; \mathbb{R})$, then $u \in C^\infty(\Omega_0; \mathbb{R}^N)$.

In section 8 we shall discuss a class of examples satisfying (H3), (H4), and in particular demonstrate the following

Theorem 3. *Assume* $2 \leqq q < \infty$. *Suppose that the* C^2 *function*

$$G: M^{n\times N} \to \mathbb{R}$$

is quasiconvex and satisfies

$$|D^2G(P)| \leqq C(1+|P|^{q-2})$$

for some constant C *and all* $P \in M^{n\times N}$. *Then*

$$F(P) \equiv a|P|^2 + b|P|^q + G(P), \quad (P \in M^{n\times N})$$

verifies (H3), (H4), *provided* $a, b > 0$.

PART I. PROOF OF THEOREM 1

3. A Caccioppoli inequality

For this section we assume that F satisfies (H1), (H2) and that $u \in W^{1,\infty}(\Omega; \mathbb{R}^N)$ is a minimizer of $I[\cdot]$. First we give a "Caccioppoli type" inequality (cf. GIAQUINTA [8, p. 76–77]).

Lemma 3.1. *There exists a constant* C_1 *such that*

$$\fint_{B(x,r/2)} |Du - A|^2 \, dy \leqq \frac{C_1}{r^2} \fint_{B(x,r)} |u - a - A(y - x)|^2 \, dy \tag{3.1}$$

for all $B(x, r) \subset \Omega$, $a \in \mathbb{R}^N$, *and* $A \in M^{n \times N}$.

Proof. We may assume $x = 0$. Let $r/2 \leqq t < s \leqq r$ and choose $\zeta \in C_0^\infty(\Omega; \mathbb{R})$ so that

$$\zeta \equiv 1 \text{ on } B(t), \quad \zeta \equiv 0 \text{ on } \Omega \setminus B(s)$$

$$0 \leqq \zeta \leqq 1, \quad |D\zeta| \leqq \frac{C}{s - t}.$$

Define

$$\phi \equiv \zeta(u - a - Ay), \quad \psi \equiv (1 - \zeta)(u - a - Ay);$$

then

$$D\phi + D\psi = Du - A. \tag{3.2}$$

Since $\zeta = 0$ on $\partial B(s)$, the hypotheses (H1) and (H2) imply

$$\begin{aligned} \int_{B(s)} [F(A) + \gamma |D\phi|^2] \, dy &\leqq \int_{B(s)} F(A + D\phi) \, dy \\ &= \int_{B(s)} F(Du - D\psi) \, dy \quad \text{by (3.2)} \\ &\leqq \int_{B(s)} [F(Du) - DF(Du) \, D\psi + C |D\psi|^2] \, dy. \end{aligned} \tag{3.3}$$

Since u is a minimizer, we have by (3.2)

$$\begin{aligned} \int_{B(s)} F(Du) \, dy &\leqq \int_{B(s)} F(Du - D\phi) \, dy = \int_{B(s)} F(A + D\psi) \, dy \\ &\leqq \int_{B(s)} [F(A) + DF(A) \, D\psi + C |D\psi|^2] \, dy. \end{aligned}$$

This inequality combined with (3.3) gives

$$\gamma \int_{B(s)} |D\phi|^2 \, dy \leqq \int_{B(s)} \{[DF(A) - DF(Du)] \, D\psi + C |D\psi|^2\} \, dy.$$

Then (H2) and the definition of ϕ imply

$$\int_{B(t)} |Du - A|^2 \, dy \leqq C \int_{B(s)} (|Du - A| \, |D\psi| + |D\psi|^2) \, dy. \tag{3.4}$$

Now $\psi \equiv 0$ on $B(t)$ and

$$\begin{aligned} |D\psi| &= |(1 - \zeta)(Du - A) - D\zeta \otimes (u - a - Ay)| \\ &\leqq C |Du - A| + \frac{C}{s - t} |u - a - Ay| \end{aligned}$$

on $B(s) \setminus B(t)$. Hence (3.4) yields

(3.5)

$$\int_{B(t)} |Du - A|^2\, dy \leqq C \int_{B(s)\setminus B(t)} |Du - A|^2\, dy + \frac{C}{(s-t)^2} \int_{B(r)} |u - a - Ay|^2\, dy.$$

We add $C \int_{B(t)} |Du|^2\, dy$ to both sides, thus obtaining

$$\int_{B(t)} |Du - A|^2\, dy \leqq \theta \int_{B(s)} |Du - A|^2\, dy + \frac{C}{(s-t)^2} \int_{B(r)} |u - a - Ay|^2\, dy$$

for

$$\theta \equiv \frac{C}{C+1} < 1.$$

This inequality is valid for $r/2 \leqq t < s \leqq r$. We may therefore apply Lemma V.3.1 of GIAQUINTA [8] (cf. also Lemma 5.2 below) to derive

$$\int_{B(r/2)} |Du - A|^2\, dy \leqq \frac{C}{r^2} \int_{B(r)} |u - a - Ay|^2\, dy,$$

as required. □

Remark. The proof from (3.5) onward uses a technique due to GIAQUINTA & GIUSTI [10].

4. Blow-up procedure

Next we adapt the blow-up technique of GIUSTI & MIRANDA [11] to obtain a basic estimate for the mean squared oscillation of Du over small balls. Since we have no control on D^2u, we must use Lemma 3.1 to modify the concluding steps of their argument.

Let us write

(4.1) $$U(x, r) \equiv \fint_{B(x,r)} |Du - (Du)_{x,r}|^2\, dy.$$

Lemma 4.1. *There exists a constant C_2 with the property that for each $0 < \tau < 1/4$ there exists $\varepsilon(\tau) > 0$ such that, for every $B(x, r) \subset \Omega$, the relation*

(4.2) $$U(x, r) \leqq \varepsilon(\tau)$$

implies

(4.3) $$U(x, \tau r) \leqq C_2 \tau^2 U(x, r).$$

Proof. Let C_2 be a constant, which will be determined later, and fix $0 < \tau < 1/4$.

Were the assertion of the lemma false, there would exist for $m = 1, 2, \ldots$ balls $B(x_m, r_m) \subset \Omega$ such that

$$U(x_m, r_m) \equiv \lambda_m^2 \to 0 \quad \text{as } m \to \infty, \tag{4.4}$$

but

$$U(x_m, \tau r_m) > C_2 \tau^2 \lambda_m^2 \quad (m = 1, 2, \ldots). \tag{4.5}$$

Define

$$\begin{aligned} a^m &\equiv (u)_{x_m, r_m}, & b^m &\equiv (u)_{x_m, 2\tau r_m}, & c^m &\equiv (u)_{x_m, \tau r_m} \\ A^m &\equiv (Du)_{x_m, r_m}, & B^m &\equiv (Du)_{x_m, 2\tau r_m}, & C^m &\equiv (Du)_{x_m, \tau r_m}, \end{aligned} \tag{4.6}$$

and then set

$$v^m(z) \equiv \frac{u(x_m + r_m z) - a^m - r_m A^m z}{\lambda_m r_m} \quad (z \in B). \tag{4.7}$$

Thus

$$Dv^m(z) = \frac{Du(x_m + r_m z) - A^m}{\lambda_m} \tag{4.8}$$

and also

$$(v^m)_1 = 0, \quad (Dv^m)_1 = 0.$$

Define

$$\begin{aligned} d^m &\equiv (v^m)_{2\tau}, & e^m &\equiv (v^m)_\tau \\ D^m &\equiv (Dv^m)_{2\tau}, & E^m &= (Dv^m)_\tau. \end{aligned} \tag{4.9}$$

Now (4.4) implies

$$\fint_B |Dv^m|^2 \, dz = 1. \tag{4.10}$$

Since $(v^m)_1 = 0$, we have

$$\fint_B |v^m|^2 \, dz \leqq C, \quad (m = 1, 2, \ldots) \tag{4.11}$$

On the other hand, (4.5) gives

$$\fint_{B(\tau)} |Dv^m - E^m|^2 \, dx > C_2 \tau^2, \quad (m = 1, 2, \ldots) \tag{4.12}$$

Now, since u is Lipschitz continuous, we have

$$|A^m| \leqq C \quad (m = 1, 2, \ldots).$$

In view of this estimate, (4.10) and (4.11), there exists a subsequence, which upon relabeling we index also by m, such that

$$\begin{cases} A^m \to A & \text{in } M^{n\times N} \\ v^m \to v & \text{strongly in } L^2(B;\mathbb{R}^N) \\ Dv^m \rightharpoonup Dv & \text{weakly in } L^2(B;M^{n\times N}) \end{cases} \tag{4.13}$$

for some $A \in M^{n\times N}$ and $v \in H^1(B;\mathbb{R}^N)$.

For future reference we set

$$\begin{aligned} d &\equiv (v)_{2\tau}, & e &\equiv (v)_\tau \\ D &\equiv (Dv)_{2\tau}, & E &\equiv (Dv)_\tau . \end{aligned} \tag{4.14}$$

We now *claim* that v is a weak solution of the linear elliptic system

$$\frac{d}{dz_\alpha}\left(\frac{\partial^2 F}{\partial p^i_\alpha\,\partial p^j_\beta}(A)\,v^j_{z_\beta}\right) = 0 \quad (i = 1, \dots, N) \tag{4.15}$$

in B. Indeed, since u is a weak solution of the Euler-Lagrange equations

$$\frac{d}{dy_\alpha}\left(\frac{\partial F}{\partial p^i_\alpha}(Du)\right) = 0 \quad (i = 1, \dots, N)$$

in Ω, we observe from (4.8) that v^m satisfies

$$\frac{1}{\lambda_m}\int_B \left[\frac{\partial F}{\partial p^i_\alpha}(\lambda_m\,Dv^m + A^m) - \frac{\partial F}{\partial p^i_\alpha}(A^m)\right]\phi^i_{z_\alpha}\,dz = 0$$

for all $\phi \in C^1_0(B;\mathbb{N})$. Hence

$$\int_B \left[\int_0^1 \frac{\partial^2 F}{\partial p^i_\alpha\,\partial p^j_\beta}(s\lambda_m\,Dv^m + A^m)\,ds\right] v^{m,j}_{z_\beta}\phi^i_{z_\alpha}\,dz = 0. \tag{4.16}$$

Since we may assume $\lambda_m\,Dv^m \to 0$ a.e., the hypothesis (H2) and (4.13) together give

$$\int_0^1 D^2F(s\lambda_m\,Dv^m + A^m)\,ds \to D^2F(A)$$

strongly in L^2. We can now pass to limit in (4.16) and complete the proof of (4.15).

We next recall from MORREY [15, Theorems 4.4.3 and 4.4.1] or FEDERER [7, Theorem 5.1.10] that (H1) implies the strong *Legendre-Hadamard condition*

$$\frac{\partial^2 F}{\partial p^i_\alpha\,\partial p^j_\beta}(A)\,\xi^i\xi^j\eta_\alpha\eta_\beta \geqq \gamma\,|\xi|^2\,|\eta|^2 \quad (\xi \in \mathbb{R}^N,\ \eta \in \mathbb{R}^n).$$

Thus from the theory of linear elliptic partial differential equations (cf. MORREY [15], GIAQUINTA [8]) it follows that v is smooth and

$$\sup_{B(1/2)} |D^2 v|^2 \leqq C \fint_B |Dv|^2\,dz \leqq C.$$

Consequently

$$\fint_{B(2\tau)} |Dv - D|^2\, dz \leqq C\tau^2,$$

and so

$$\lim_{m\to\infty} \fint_{B(2\tau)} |v^m - d^m - D^m z|^2\, dz = \fint_{B(2\tau)} |v - d - Dz|^2\, dz$$

$$\leqq C\tau^2 \fint_{B(2\tau)} |Dv - D|^2\, dz \leqq C\tau^4. \tag{4.17}$$

But Lemma 3.1 provides the estimate

$$\fint_{B(x_m,\tau r_m)} |Du - B^m|^2\, dy \leqq \frac{C_1}{(2\tau r_m)^2} \fint_{B(x_m,2\tau r_m)} |u - b^m - B^m(y - x_m)|^2\, dy,$$

which upon dividing by λ_m^2 and rescaling gives

$$\fint_{B(\tau)} |Dv^m - D^m|^2\, dz \leqq \frac{C_1}{\tau^2} \fint_{B(2\tau)} |v^m - d^m - D^m z|^2\, dz.$$

This and (4.17) imply

$$\limsup_{m\to\infty} \fint_{B(\tau)} |Dv^m - E^m|^2\, dz \leqq \limsup_{m\to\infty} \fint_{B(\tau)} |Dv^m - D^m|^2\, dz$$

$$\leqq \lim_{m\to\infty} \frac{C}{\tau^2} \fint_{B(2\tau)} |v^m - d^m - D^m z|^2\, dz \leqq C_3\tau^2,$$

contradicting (4.12) provided we choose $C_2 > C_3$. □

Proof of Theorem 1. This is now a routine consequence of Lemma 4.1; see GIAQUINTA [8, p. 95–96] or the analogous argument in section 7 below. □

PART II. PROOF OF THEOREM 2

5. A Caccioppoli inequality

We devote the next three sections to a proof of Theorem 2, following the general lines of the proof of Theorem 1. For these sections we therefore suppose $2 \leqq q < \infty$, the function F satisfies (H3), (H4), and $u \in W^{1,q}(\Omega; \mathbb{R}^N)$ is a minimizer of $I[\cdot]$.

Lemma 5.1. *For each $L > 0$ there exists a constant $C_1(L)$ such that*

(5.1)

$$\fint_{B(x,r/2)} (1 + |Du - A|^{q-2})\, |Du - A|^2\, dy$$

$$\leqq C_1(L) \left(\frac{1}{r^2} \fint_{B(x,r)} |u - a - A(y - x)|^2\, dy + \frac{1}{r^q} \fint_{B(x,r)} |u - a - A(y - x)|^q\, dy \right)$$

for all $B(x,r) \subset \Omega$, $a \in \mathbb{R}^N$, *and* $A \in M^{n\times N}$ *with*

$$|A| \leqq L. \tag{5.2}$$

Proof. We may assume $x = 0$. Let $r/2 \leqq t < s \leqq r$ and choose $\zeta \in C_0^\infty(\Omega; \mathbb{R})$ satisfying

$$\zeta \equiv 1 \text{ on } B(t), \quad \zeta \equiv 0 \text{ on } \Omega \setminus B(s)$$

$$0 \leqq \zeta \leqq 1, \quad |D\zeta| \leqq \frac{C}{s-t}.$$

Define

$$\phi \equiv \zeta(u - a - Ay), \quad \psi \equiv (1-\zeta)(u - a - Ay);$$

then

$$D\phi + D\psi = Du - A. \tag{5.3}$$

Since $\zeta = 0$ on $\partial B(s)$, the hypotheses (H3) and (H4) imply

$$\begin{aligned}
\int_{B(s)} [F(A) + \gamma(1 + |D\phi|^{q-2})\,|D\phi|^2]\,dy
&\leqq \int_{B(s)} F(A + D\phi)\,dy \\
&= \int_{B(s)} F(Du - D\psi)\,dy \quad \text{by (5.3)} \\
&\leqq \int_{B(s)} [F(Du) - DF(Du)\,D\psi + C(1 + |Du|^{q-2} + |D\psi|^{q-2})\,|D\psi|^2]\,dy.
\end{aligned} \tag{5.4}$$

Since u is a minimizer, we have

$$\begin{aligned}
\int_{B(s)} F(Du)\,dy &\leqq \int_{B(s)} F(Du - D\phi)\,dy \\
&= \int_{B(s)} F(A + D\psi)\,dy \quad \text{by (5.3)} \\
&\leqq \int_{B(s)} [F(A) + DF(A)\,D\psi + C(1 + |D\psi|^{q-2})\,|D\psi|^2]\,dy,
\end{aligned}$$

according to (H4) and (5.2). This inequality combined with (5.4) gives

(5.5)

$$\begin{aligned}
&\gamma \int_{B(s)} (1 + |D\phi|^{q-2})\,|D\phi|^2\,dy \\
&\leqq \int_{B(s)} \{[DF(A) - DF(Du)]\,D\psi + C(1 + |Du|^{q-2} + |D\psi|^{q-2})\,|D\psi|^2\}\,dy \\
&\leqq C \int_{B(s)} [(1 + |Du|^{q-2})\,|Du - A|\,|D\psi| + (1 + |Du|^{q-2} + |D\psi|^{q-2})\,|D\psi|^2]\,dy,
\end{aligned}$$

where we have once again used (H4) and (5.2).

Now (5.2) implies

$$|Du|^{q-2} \leqq C(1 + |Du - A|^{q-2}). \tag{5.6}$$

Furthermore $D\phi = Du - A$ on $B(t)$ and $\psi \equiv 0$ on $B(t)$. Recalling the elementary inequalities

$$\alpha^{q-1}\beta \leqq \alpha^q + \beta^q, \quad \alpha^{q-2}\beta^2 \leqq \alpha^q + \beta^q \quad (\alpha, \beta \geqq 0), \tag{5.7}$$

and inequality (5.5), we obtain the estimate

$$\int_{B(t)} (1 + |Du - A|^{q-2}) |Du - A|^2 \, dy$$
$$\leqq C \int_{B(s)\setminus B(t)} [(1 + |Du - A|^{q-2}) |Du - A|^2 + (1 + |D\psi|^{q-2}) |D\psi|^2] \, dy. \tag{5.8}$$

Since

$$|D\psi| \leqq C |Du - A| + \frac{C}{(s-t)} |u - a - Ay|,$$

from (5.8) it follows that

$$\int_{B(t)} (1 + |Du - A|^{q-2}) |Du - A|^2 \, dy$$
$$\leqq C \int_{B(s)\setminus B(t)} (1 + |Du - A|^{q-2}) |Du - A|^2 \, dy$$
$$+ C \int_{B(r)} \left[\frac{|u - a - Ay|^2}{(s-t)^2} + \frac{|u - a - Ay|^q}{(s-t)^q} \right] dy.$$

Consequently

$$\int_{B(t)} (1 + |Du - A|^{q-2}) |Du - A|^2 \, dy$$
$$\leqq \theta \int_{B(s)} (1 + |Du - A|^{q-2}) |Du - A|^2 \, dy$$
$$+ \frac{C}{(s-t)^2} \int_{B(r)} |u - a - Ay|^2 \, dy + \frac{C}{(s-t)^q} \int_{B(r)} |u - a - Ay|^q \, dy$$

for

$$\theta = \frac{C}{C+1} < 1.$$

We now use the following lemma to complete the proof of (5.1). □

Lemma 5.2. *Let*

$$f \colon [r/2, r] \to [0, \infty)$$

be bounded and satisfy

$$f(t) \leqq \theta f(s) + \frac{A}{(s-t)^2} + \frac{B}{(s-t)^q} \tag{5.9}$$

for some $\theta < 1$ *and all* $r/2 \leqq t < s \leqq r$.

Then there exists a constant $C = C(\theta, q)$ *such that*

$$f\left(\frac{r}{2}\right) \leqq C\left(\frac{A}{r^2} + \frac{B}{r^q}\right). \tag{5.10}$$

Remark. This is a variant of as result of GIAQIUNTA & GIUSTI [10], whose proof we adapt to the case $q > 2$.

Proof. Set

$$t_k \equiv r\left(1 - \frac{\tau^k}{2}\right) \quad (k = 0, 1, \ldots)$$

where τ (<1) is a constant which will be selected later. Then

$$t_0 = \frac{r}{2}, \quad \lim_{k\to\infty} t_k = r, \quad t_{k+1} - t_k = \frac{r}{2}(1-\tau)\,\tau^k.$$

According to (5.9) we have

$$\begin{aligned} f(t_k) &\leqq \theta f(t_{k+1}) + \frac{A}{(t_{k+1} - t_k)^2} + \frac{B}{(t_{k+1} - t_k)^q} \\ &\leqq \theta f(t_{k+1}) + \frac{C}{\tau^{kq}}\left(\frac{A}{r^2} + \frac{B}{r^q}\right) \quad (k = 0, 1, \ldots). \end{aligned}$$

Iterate this inequality to obtain

$$f(t_0) \leqq \theta^k f(t_k) + C\left(\frac{A}{r^2} + \frac{B}{r^q}\right)\sum_{l=0}^{k-1}\left(\frac{\theta}{\tau^q}\right)^l.$$

Choose $\tau < 1$ so that $\theta < \tau^q < 1$ and then send k to infinity.

This completes the proof. □

6. Blow-up

We next modify the argument of section 4 for $q \geqq 2$. The main difficulties here arise from the nonhomogeneity of estimate (5.1).

Define

$$U(x, r) \equiv \fint_{B(x,r)} (1 + |Du - (Du)_{x,r}|)^{q-2}\,|Du - (Du)_{x,r}|^2\,dy. \tag{6.1}$$

Lemma 6.1. *For each* $L > 0$ *there exists a constant* $C_2(L)$ *with the property that for each* $0 < \tau < 1/4$ *there exists* $\varepsilon(L, \tau) > 0$ *such that for every* $B(x, r) \subset \Omega$,

$$|(Du)_{x,r}| \leqq L, \quad |(Du)_{x,\tau r}| \leqq L, \tag{6.2}$$

and

$$U(x, r) \leqq \varepsilon(L, \tau) \tag{6.3}$$

imply

$$U(x, \tau r) \leqq C_2(L)\, \tau^2 U(x, r). \tag{6.4}$$

Proof. Given $L > 0$, we let $C_2(L) > 0$ be a constant which we shall determine later. Also fix $0 < \tau < 1/4$.

Were the assertion of the lemma false, there would exist balls $B(x_m, r_m) \subset \Omega$, $m = 1, 2, \ldots,$ such that

$$|(Du)_{x_m, r_m}| \leqq L, \quad |(Du)_{x_m, \tau r_m}| \leqq L, \tag{6.5}$$

$$U(x_m, r_m) \equiv \lambda_m^2 \to 0 \quad \text{as } m \to \infty, \tag{6.6}$$

but

$$U(x_m, \tau r_m) > C_2(L)\, \tau^2 \lambda_m^2 \quad (m = 1, 2, \ldots). \tag{6.7}$$

Define

$$\begin{aligned} a^m &\equiv (u)_{x_m, r_m}, & b^m &\equiv (u)_{x_m, 2\tau r_m}, & c^m &\equiv (u)_{x_m, \tau r_m}, \\ A^m &\equiv (Du)_{x_m, r_m}, & B^m &\equiv (Du)_{x_m, 2\tau r_m}, & C^m &\equiv (Du)_{x_m, \tau r_m}, \end{aligned} \tag{6.8}$$

and then set

$$v^m(z) \equiv \frac{u(x_m + r_m z) - a^m - r_m A^m z}{\lambda_m r_m} \quad (z \in B). \tag{6.9}$$

Thus

$$Dv^m(z) = \frac{Du(x_m + r_m z) - A^m}{\lambda_m} \tag{6.10}$$

and

$$(v^m)_1 = 0, \quad (Dv^m)_1 = 0.$$

Set

$$\begin{aligned} d^m &\equiv (v^m)_{2\tau}, & e^m &\equiv (v^m)_\tau \\ D^m &\equiv (Dv^m)_{2\tau}, & E^m &\equiv (Dv^m)_\tau. \end{aligned} \tag{6.11}$$

Now (6.6) implies

$$\fint_B (1 + \lambda_m^{q-2} |Dv^m|^{q-2}) |Dv^m|^2 \, dz = 1 \tag{6.12}$$

and, since $(v^m)_1 = 0$, we also have

$$\fint_B |v^m|^2 \, dz \leqq C \quad (m = 1, 2, \ldots). \tag{6.13}$$

Furthermore (6.7) gives

(6.14)

$$\fint_{B(\tau)} (1 + \lambda_m^{q-2} |Dv^m - E^m|^{q-2}) |Dv^m - E^m|^2 \, dz > C_2(L)\, \tau^2 \quad (m = 1, 2, \ldots).$$

Now in view of (6.5)

$$|A^m| \leqq L \quad (m = 1, 2, \ldots).$$

This bound, together with (6.12) and (6.13), implies the existence of a subsequence, which upon relabeling we continue to index by m, such that

$$(6.15) \qquad \begin{cases} A^m \to A & \text{in } M^{n\times N} \\ v^m \to v & \text{strongly in } L^2(B; \mathbb{R}^N) \\ Dv^m \rightharpoonup Dv & \text{weakly in } L^2(B; M^{n\times N}) \end{cases}$$

for some $A \in M^{n\times N}$ and $v \in H^1(B; \mathbb{R}^N)$, with

$$(6.16) \qquad |A| \leqq L.$$

For future reference, set

$$(6.17) \qquad \begin{aligned} d &\equiv (v)_{2\tau}, & e &\equiv (v)_\tau \\ D &\equiv (Dv)_{2\tau}, & E &\equiv (Dv)_\tau . \end{aligned}$$

In view of Lemma 6.2 below, v is a weak solution of the linear elliptic system

$$\frac{d}{dz_\alpha}\left(\frac{\partial^2 F}{\partial p_\alpha^i\, \partial p_\beta^j}(A)\, v_{z_\beta}^j\right) = 0 \quad (i = 1, \ldots, N)$$

in B. Furthermore, hypothesis (H3) implies (H1), which in turn, as seen in section 4, gives the strong Legendre-Hadamard condition

$$\frac{\partial^2 F}{\partial p_\alpha^i\, \partial p_\beta^j}(A)\, \xi^i \xi^j \eta_\alpha \eta_\beta \geqq \gamma\, |\xi|^2\, |\eta|^2 \quad (\xi \in \mathbb{R}^N, \eta \in \mathbb{R}^n).$$

Finally note from (H4) and (6.16) that

$$|D^2F(A)| \leqq C(1 + |L|^{q-2}).$$

Consequently, standard estimates for linear elliptic partial differential equations show v to be smooth and provide the estimate

$$\sup_{B(1/2)} |D^2 v|^2 \leqq C(L) \fint_B |Dv|^2\, dz \leqq C(L).$$

Hence

$$(6.18) \qquad \fint_{B(2\tau)} |Dv - E|^2\, dz \leqq C(L)\, \tau^2;$$

the reader should note carefully here (cf. (6.17)) that E is the average of Dv over $B(\tau)$, not $B(2\tau)$.

Therefore

$$(6.19) \qquad \begin{aligned} \lim_{m\to\infty} \fint_{B(2\tau)} |v^m - d^m - E^m z|^2\, dz &= \fint_{B(2\tau)} |v - d - Ez|^2\, dz \\ &\leqq C\tau^2 \fint_{B(2\tau)} |Dv - E|^2\, dz \leqq C(L)\, \tau^4 . \end{aligned}$$

We next *claim* that if $q > 2$, then

$$\lim_{m\to\infty} \lambda_m^{q-2} \fint_{B(2\tau)} |v^m - d^m - E^m z|^q \, dz = 0. \tag{6.20}$$

To see this, fix any $q < r$ and set

$$q^* \equiv \begin{cases} \dfrac{qn}{n-q} & \text{if } 2 < q < n \\ r & \text{if } n \leqq q < \infty. \end{cases}$$

Then $2 < q < q^*$ and so

$$\frac{1}{q} = \frac{1-\alpha}{2} + \frac{\alpha}{q^*}$$

for some $0 < \alpha < 1$. Consequently

$$\begin{aligned} &\lambda_m^{q-2} \fint_{B(2\tau)} |v^m - d^m - E^m z|^q \, dz \\ &\quad \leqq \lambda_m^{q-2} \left(\fint_{B(2\tau)} |v^m - d^m - E^m z|^2 \, dz \right)^{\frac{(1-\alpha)q}{2}} \left(\fint_{B(2\tau)} |v^m - d^m - E^m z|^{q^*} \, dz \right)^{\frac{\alpha q}{q^*}} \\ &\quad \leqq C(L, \tau) \, \lambda_m^{q-2} \left(\fint_{B(2\tau)} |v^m - d^m - E^m z|^{q^*} \, dz \right)^{\frac{\alpha q}{q^*}} \quad \text{by (6.19)} \\ &\quad \leqq C(L, \tau) \, \lambda_m^{q-2} \left(\fint_{B(2\tau)} |Dv^m - E^m|^q \, dz \right)^{\alpha}, \end{aligned} \tag{6.21}$$

according to the Sobolev-Poincaré inequality.

Now

$$|E^m|^q = \left| \fint_{B(\tau)} Dv^m \, dz \right|^q \leqq C \fint_{B(2\tau)} |Dv^m|^q \, dz$$

and so

$$\fint_{B(2\tau)} |Dv^m - E^m|^q \, dz \leqq C(\tau) \fint_{B(2\tau)} |Dv^m|^q \, dz.$$

Insert this inequality into (6.21) to obtain

$$\begin{aligned} \lambda_m^{q-2} \fint_{B(2\tau)} |v^m - d^m - E^m z|^q \, dz &\leqq C(L, \tau) \, \lambda_m^{q-2} \left(\fint_B |Dv^m|^q \, dz \right)^{\alpha} \\ &\leqq C(L, \tau) \, \lambda_m^{(q-2)(1-\alpha)} \left(\fint_B \lambda_m^{q-2} |Dv^m|^q \, dz \right)^{\alpha} \\ &\leqq C(L, \tau) \, \lambda_m^{(q-2)(1-\alpha)} \end{aligned}$$

by (6.12). The last term goes to zero as $m \to \infty$, since $\lambda_m \to 0$, $0 < \alpha < 1$ and $q > 2$, proving (6.20).

We now recall Lemma 5.1, which provides the estimate

$$\fint_{B(x_m,\tau r_m)} (1 + |Du - C^m|^{q-2})\,|Du - C^m|^2\,dy$$
$$\leqq C_1(L)\left(\frac{1}{(2\tau r_m)^2}\fint_{B(x_m,2\tau r_m)} |u - b^m - C^m(y - x_m)|^2\,dy + \frac{1}{(2\tau r_m)^q}\fint_{B(x_m,2\tau r_m)} |u - b^m - C^m(y - x_m)|^q\,dy\right);$$

note from (6.5) and (6.8) that $|C^m| \leqq L$, as required. We divide the inequality above by λ_m^2 and rescale to obtain

$$\fint_{B(\tau)} (1 + \lambda_m^{q-2}\,|Dv^m - E^m|^{q-2})\,|Dv^m - E^m|^2\,dz$$
$$\leqq C(L)\left(\frac{1}{\tau^2}\fint_{B(2\tau)} |v^m - d^m - E^m z|^2\,dz + \frac{\lambda_m^{q-2}}{\tau^q}\fint_{B(2\tau)} |v^m - d^m - E^m z|^q\,dz\right).$$

In view of (6.19) and (6.20) we have

$$\limsup_{m\to\infty} \fint_{B(\tau)} (1 + \lambda_m^{q-2}\,|Dv^m - E^m|^{q-2})\,|Dv^m - E^m|^2\,dz \leqq C_3(L)\,\tau^2,$$

contradicting (6.14) provided we choose $C_2(L) > C_3(L)$. □

Lemma 6.2. *The function* $v \in H^1(B;\mathbb{R}^N)$ *is a weak solution of the linear elliptic system*

$$\frac{d}{dz_\alpha}\left(\frac{\partial^2 F}{\partial p_\alpha^i\,\partial p_\beta^j}(A)\,v_{z_\beta}^j\right) = 0 \quad (i = 1, \ldots, N)$$

in B.

Proof. It is not difficult to prove from (H4) that u is a weak solution of the Euler-Lagrange equations

$$\frac{d}{dy_\alpha}\left(\frac{\partial F}{\partial p_\alpha^i}(Du)\right) = 0 \quad (i = 1, \ldots, N)$$

in Ω. Thus, for each $\phi \in C_0^1(B;\mathbb{R}^N)$, the relation (6.10) implies

$$\begin{aligned} 0 &= \frac{1}{\lambda_m}\int_B \left[\frac{\partial F}{\partial p_\alpha^i}(\lambda_m\,Dv^m + A^m) - \frac{\partial F}{\partial p_\alpha^i}(A^m)\right]\phi_{z_\alpha}^i\,dz \\ &= \int_B \left[\int_0^1 \frac{\partial^2 F}{\partial p_\alpha^i\,\partial p_\beta^j}(s\lambda_m\,Dv^m + A^m)\,ds\right] v_{z_\beta}^{m,j}\phi_{z_\alpha}^i\,dz \\ &= \int_B \frac{\partial^2 F}{\partial p_\alpha^i\,\partial p_\beta^j}(A^m)\,v_{z_\beta}^{m,j}\phi_{z_\alpha}^i\,dz \\ &\quad + \int_B \left[\int_0^1 \frac{\partial^2 F}{\partial p_\alpha^i\,\partial p_\beta^j}(s\lambda_m\,Dv^m + A^m) - \frac{\partial^2 F}{\partial p_\alpha^i\,\partial p_\beta^j}(A^m)\,ds\right] v_{z_\beta}^{m,j}\phi_{z_\alpha}^i\,dz \equiv I + II. \end{aligned} \tag{6.22}$$

Clearly (6.15) implies

$$I \to \int_B \frac{\partial^2 F}{\partial p_\alpha^i \, \partial p_\beta^j}(A)\, v_{z_\beta}^j \phi_{z_\alpha}^i \, dz \quad \text{as } m \to \infty. \tag{6.23}$$

We furthermore claim that

$$II \to 0 \quad \text{as } m \to \infty. \tag{6.24}$$

To see this, fix $\varepsilon > 0$. Owing to (6.15), we may assume $\lambda_m Dv^m \to 0$ a.e. as $m \to \infty$. Hence there exists a measurable subset $E \subset B$ such that

$$|E| < \varepsilon^2 \tag{6.25}$$

and

$$\lambda_m Dv^m \to 0 \quad \text{uniformly on } B \setminus E. \tag{6.26}$$

From hypothesis (H4) we obtain the estimate

$$\begin{aligned} |II| &\leqq \int_{B\setminus E} \int_0^1 |D^2F(s\lambda_m Dv^m + A^m) - D^2F(A^m)| \, ds \, |Dv^m| \, |D\phi| \, dz \\ &\quad + C \int_E (1 + \lambda_m^{q-2} |Dv^m|^{q-2}) \, |Dv^m| \, |D\phi| \, dz \equiv A + B. \end{aligned}$$

Using (6.26), we see that

$$A \leqq \varepsilon \left(\int_B |Dv^m|^2 \, dz \right)^{1/2} \leqq \varepsilon \quad \text{for } m \geqq m_1(\varepsilon).$$

On the other hand, if $q > 2$,

$$\begin{aligned} B &\leqq C \int_E |Dv^m| \, dz + C\lambda_m^{q-2} \int_E |Dv^m|^{q-1} \, dz \\ &\leqq C \, |E|^{1/2} \left(\int_B |Dv^m|^2 \, dz \right)^{1/2} + C\lambda_m^{q-2} \int_B (C(\varepsilon) + \varepsilon \, |Dv^m|^q) \, dz \\ &\leqq C\varepsilon + C(\varepsilon)\, \lambda_m^{q-2} + \varepsilon \int_B \lambda_m^{q-2} \, |Dv^m|^q \, dz \\ &\leqq C\varepsilon + C(\varepsilon)\, \lambda_m^{q-2} \quad \text{by (6.12)} \\ &\leqq C\varepsilon \quad \text{for } m \geqq m_2(\varepsilon). \qquad \square \end{aligned}$$

7. Proof of Theorem 2

Lemma 7.1. *For each* $L > 0$ *and each* τ *satisfying*

$$0 < \tau < C_2(2L)^{-1/2} \tag{7.1}$$

there exists a number $\eta(L, \tau) > 0$ *such that, for every* $B(x, r) \subset \Omega$, *the inequalities*

$$|(Du)_{x,r}| \leqq L, \quad |(Du)_{x,\tau r}| \leqq L, \tag{7.2}$$

and

$$U(x, r) \leqq \eta(L, \tau) \tag{7.3}$$

imply

$$U(x, \tau^l r) \leqq C_2(2L)\, \tau^2 U(x, r) \qquad (l = 1, 2, \ldots). \tag{7.4}$$

Here $C_2(2L)$ is the constant from Lemma 6.1, with $2L$ replacing L, and $U(x, r)$ is defined by (6.1).

Proof. Given $L > 0$, we fix $0 < \tau < C_2(2L)^{-1/2}$ and then define

$$\eta(L, \tau) \equiv \min\left\{\varepsilon(2L, \tau), \frac{\tau^{2n} L^2}{4}\,[1 - \sqrt{C_2(2L)}\,\tau]^2\right\}, \tag{7.5}$$

where $\varepsilon(2L, \tau)$ is the constant from Lemma 6.1, with $2L$ replacing L.

Assume that (7.2) and (7.3) hold. We shall prove by induction that (7.4) is valid for $l = 1, 2, \ldots$ The case $l = 1$ is immediate from Lemma 6.1, since $U(x, r) \leqq \eta(L, \tau) \leqq \varepsilon(2L, \tau)$.

Now assume that (7.4) holds for $l = 1, \ldots, k$. We claim that

$$|(Du)_{x,\tau^k r}| \leqq 2L, \tag{7.6}$$

$$|(Du)_{x,\tau^{k+1} r}| \leqq 2L, \tag{7.7}$$

$$U(x, \tau^k r) \leqq \varepsilon(2L, \tau). \tag{7.8}$$

Once these relations are verified, Lemma 6.1 (with $2L$ replacing L and $\tau^k r$ replacing r) yields

$$U(x, \tau^{k+1} r) \leqq C_2(2L)\, \tau^2 U(x, \tau^k r) \leqq (C_2(2L)\, \tau^2)^{k+1}\, U(x, r),$$

and this proves (7.4) for $l = k + 1$.

Proof of (7.6). For $l = 0, 1, \ldots$ we have

$$\begin{aligned} |(Du)_{x,\tau^{l+1} r} - (Du)_{x,\tau^l r}| &\leqq \fint_{B(x,\tau^{l+1} r)} |Du - (Du)_{x,\tau^l r}|\, dy \\ &\leqq \frac{1}{\tau^n}\left(\fint_{B(x,\tau^l r)} |Du - (Du)_{x,\tau^l r}|^2\, dy\right)^{\frac{1}{2}} \leqq \frac{1}{\tau^n}\, U(x, \tau^l r)^{\frac{1}{2}}. \end{aligned} \tag{7.9}$$

Consequently,

$$\begin{aligned} |(Du)_{x,\tau^k r}| &\leqq \sum_{l=1}^{k-1} |(Du)_{x,\tau^{l+1} r} - (Du)_{x,\tau^l r}| + |(Du)_{x,r}| \\ &\leqq \frac{1}{\tau^n} \sum_{l=1}^{k-1} U(x, \tau^l r)^{1/2} + L && \text{by (7.9), (7.2)} \\ &\leqq \frac{1}{\tau^n} \sum_{l=1}^{k-1} [(C_2(2L)\, \tau^2)^l\, U(x, r)]^{1/2} + L \\ &\leqq \frac{1}{\tau^n}\, \eta(L, \tau)^{1/2} (1 - \sqrt{C_2(2L)}\, \tau)^{-1} + L && \text{by (7.3)} \\ &\leqq \frac{3L}{2} && \text{by (7.5).} \end{aligned} \tag{7.10}$$

Proof of (7.7). According to (7.9),

$$\begin{aligned}|(Du)_{x,\tau^{k+1}r} - (Du)_{x,\tau^k r}| &\leqq \frac{1}{\tau^n} U(x, \tau^k r)^{1/2} \\ &\leqq \frac{1}{\tau^n} (C_2(2L)\,\tau^2)^{k/2}\, U(x, r)^{1/2} \\ &\leqq \frac{\eta(L,\tau)^{1/2}}{\tau^n} \quad \text{by (7.3)} \\ &\leqq \frac{L}{2} \quad \text{by (7.5).}\end{aligned}$$

Hence (7.10) implies

$$|(Du)_{x,\tau^{k+1}r}| \leqq |(Du)_{x,\tau^{k+1}r} - (Du)_{x,\tau^k r}| + |(Du)_{x,\tau^k r}| \leqq 2L.$$

Proof of (7.8). From (7.3) and (7.5) it follows that

$$\begin{aligned}U(x, \tau^k r) &\leqq (C_2(2L)\,\tau^2)^k\, U(x, r) \\ &\leqq U(x, r) \leqq \eta(L, \tau) \leqq \varepsilon(2L, \tau). \qquad \Box\end{aligned}$$

The basic idea of this proof is taken from GIUSTI & MIRANDA [11, Lemma 6].

Proof of Theorem 2. Set

$$\Omega_0 \equiv \left\{x \in \Omega : \lim_{r \searrow 0} (Du)_{x,r} = Du(x),\ \lim_{r \searrow 0} \fint_{B(x,r)} |Du - (Du)_{x,r}|^q\, dy = 0\right\}.$$

Since $u \in W^{1,q}(\Omega; \mathbb{R}^N)$ it follows that $|\Omega \setminus \Omega_0| = 0$.

We shall prove that Ω_0 is open and that $Du \in C^\alpha(\Omega_0)$ for each $0 < \alpha < 1$. Indeed, for each $x \in \Omega_0$ there exists a number $L = L(x)$ such that

$$(7.11) \qquad |(Du)_{s,x}| < L \quad \text{for all} \quad 0 < s < \operatorname{dist}(x, \partial\Omega).$$

Fix α, $0 < \alpha < 1$, and then select τ, $0 < \tau < C_2(2L)^{-1/2}$, so that

$$(7.12) \qquad C_2(2L)\,\tau^{2-2\alpha} \leqq 1.$$

Next choose r, $0 < r < \operatorname{dist}(x, \partial\Omega)$, such that

$$\fint_{B(x,r)} |Du - (Du)_{x,r}|^q\, dy < \left(\frac{\eta(L,\tau)}{2}\right)^{q/2};$$

this is possible since $x \in \Omega_0$. Then

$$\begin{aligned}U(x, r) &= \fint_{B(x,r)} |Du - (Du)|^2\, dy + \fint_{B(x,r)} |Du - (Du)_{x,r}|^q\, dy \\ (7.13) \qquad &\leqq \left(\fint_{B(x,r)} |Du - (Du)|^q\, dy\right)^{2/q} + \fint_{B(x,r)} |Du - (Du)_{x,r}|^q\, dy \\ &< \frac{\eta(L,\tau)}{2} + \left(\frac{\eta(L,\tau)}{2}\right)^{q/2} \leqq \eta(L, \tau).\end{aligned}$$

Also (7.11) implies

$$|(Du)_{x,r}| < L, \quad |(Du)_{x,\tau r}| < L. \tag{7.14}$$

Furthermore, since the mappings

$$x \mapsto U(x, r), \quad (Du)_{x,r}, \quad (Du)_{x,\tau r}$$

are continuous, we have

$$U(z, r) < \eta(L, \tau) \tag{7.15}$$

and

$$|(Du)_{z,r}| < L, \quad |(Du)_{z,\tau r}| < L \tag{7.16}$$

for each $z \in B(x, s)$ for some $s > 0$. Consequently Lemma 7.1 implies

$$U(z, \tau^l r) \leqq (C_2(2L)\,\tau^2)^l\, U(z, r) \quad (l = 1, 2, \ldots)$$

for each $z \in B(x, s)$. In view of (7.12), we have

$$U(z, \tau^l r) \leqq \tau^{2l\alpha} U(z, r) \leqq (\tau^l r)^{2\alpha}\, C(r, \tau) \tag{7.17}$$

for $l = 1, 2, \ldots$ and $z \in B(x, s)$. Finally, since

$$U(z, t) \geqq \fint_{B(z,t)} |Du - (Du)_{z,t}|^2\, dy,$$

the estimate (7.17) and the standard theory of partial differential equations (cf. GIAQUINTA [8, p. 70–72]) imply

$$Du \in C^\alpha(B(x, s/2)).$$

In particular, therefore, $B(x, s/2) \subset \Omega_0$ and so Ω_0 is open. □

PART III. PROOF OF THEOREM 3

8. A class of examples

In this section we shall consider a fairly broad class of non-convex examples satisfying hypotheses (H3) and (H4) of the partial regularity Theorem 2, as well as the existence hypotheses described in section 1. We assume henceforth $2 < q < \infty$.

Lemma 8.1. *There exists a constant* $\sigma > 0$ *such that*

$$\sigma(|A|^{q-2} + |B|^{q-2}) \leqq \int_0^1 (1 - s)\, |A + sB|^{q-2}\, ds \tag{8.1}$$

for all $A, B \in M^{n\times N}$.

Proof. Let Λ be a positive constant, which will be explicitly determined later. We treat two cases.

Case 1. $|B| \geqq \Lambda |A|$. We have

$$\int_0^1 (1-s)\,|A+sB|^{q-2}\,dx \geqq \tfrac{1}{8} \min_{1/2 \leqq s \leqq 1} |A+sB|^{q-2}. \tag{8.2}$$

Now, for $1/2 \leqq s \leqq 1$, we have

$$\begin{aligned} |B|^{q-2} &\leqq C\,|sB|^{q-2} \leqq C(|A+sB|^{q-2} + |A|^{q-2}) \\ &\leqq C\left(|A+sB|^{q-2} + \frac{|B|^{q-2}}{\Lambda^{q-2}}\right). \end{aligned} \tag{8.3}$$

Since $q > 2$ we may choose Λ so large that

$$\frac{C}{\Lambda^{q-2}} \leqq \tfrac{1}{2}. \tag{8.4}$$

Then (8.3) implies

$$|B|^{q-2} \leqq C\,|A+sB|^{q-2} \qquad (1/2 \leqq s \leqq 1),$$

and so (8.12) gives

$$\int_0^1 (1-s)\,|A+sB|^{q-2}\,ds \geqq C\,|B|^{q-2} \geqq \sigma(|A|^{q-2} + |B|^{q-2})$$

for some $\sigma > 0$, since $|B| \geqq \Lambda |A|$.

Case 2. $|B| \leqq \Lambda |A|$. We have

$$\int_0^1 (1-s)\,|A+sB|^{q-2}\,ds \geqq \lambda(1-\lambda) \min_{0 \leqq s \leqq \lambda} |A+sB|^{q-2}, \tag{8.5}$$

where λ, $0 < \lambda < 1$, will be chosen later. Then for $0 \leqq s \leqq \lambda$,

$$\begin{aligned} |A|^{q-2} &\leqq C(|A+sB|^{q-2} + |sB|^{q-2}) \leqq C(|A+sB|^{q-2} + \lambda^{q-2}\,|B|^{q-2}) \\ &\leqq C(|A+sB|^{q-2} + \lambda^{q-2}\Lambda^{q-2}\,|A|^{q-2}). \end{aligned} \tag{8.6}$$

Fix λ so small that

$$C\lambda^{q-2}\Lambda^{q-2} \leqq 1/2;$$

then (8.6) implies

$$|A|^{q-2} \leqq C\,|A+sB|^{q-2} \qquad (0 \leqq s \leqq \lambda).$$

Thus (8.5) yields

$$\int_0^1 (1-s)\,|A+sB|^{q-2}\,ds \geqq C\,|A|^{q-2} \geqq \sigma(|A|^{q-2} + |B|^{q-2})$$

for some $\sigma > 0$, since $|B| \leqq \Lambda |A|$. $\square$

Lemma 8.2. *There exists* $\varkappa > 0$ *such that*

$$\int_{B(x,r)} (|A|^q + \varkappa\,|D\phi|^q)\,dy \leqq \int_{B(x,r)} |A + D\phi|^q\,dy \tag{8.7}$$

for all $x \in \mathbb{R}^n$, $r > 0$, $A \in M^{n\times N}$ *and* $\phi \in C_0^1(B(x,r);\mathbb{R}^N)$.

Proof. Define

$$H(P) \equiv |P|^q \quad (P \in M^{n\times N});$$

then

$$\frac{\partial H}{\partial p_\alpha^i}(P) = q\,|P|^{q-2}\,p_\alpha^i \tag{8.8}$$

and

$$\frac{\partial^2 H}{\partial p_\alpha^i\,\partial p_\beta^j}(P) = q\,|P|^{q-2}\left(\delta_{ij}\delta_{\alpha\beta} + (q-2)\frac{p_\alpha^i p_\beta^j}{|P|^2}\right), \tag{8.9}$$

for $P \in M^{n\times N}$, $1 \leqq \alpha,\ \beta \leqq n$, and $1 \leqq i,\ j \leqq N$. Fix any $A, P \in M^{n\times N}$ and set

$$h(s) \equiv H(A + sP) \quad (0 \leqq s \leqq 1).$$

Then

$$h'(s) = DH(A + sP)\,P, \tag{8.10}$$

$$h''(s) = P^T D^2 H(A + sP)\,P, \tag{8.11}$$

and

$$h(1) = h(0) + h'(0) + \int_0^1 (1-s)\,h''(s)\,ds. \tag{8.12}$$

Using (8.9), (8.11), we see that

$$\int_0^1 (1-s)\,h''(s)\,ds \geqq q\,|P|^2 \int_0^1 (1-s)\,|A + sP|^{q-2}\,ds \geqq \varkappa\,|P|^q$$

for some $\varkappa > 0$, according to Lemma 8.1. Hence setting $P \equiv D\phi$ in (8.12) and recalling (8.8) and (8.10) gives

$$|A + D\phi|^q \geqq |A|^q + q\,|A|^{q-2}\,A\,D\phi + \varkappa\,|D\phi|^q.$$

Integrating over $B(x,r)$ and recalling that $\phi = 0$ on $\partial B(x,r)$ completes the proof. □

Proof of Theorem 3. Assume that G is quasiconvex and that

$$|D^2G(P)| \leqq C(1 + |P|^{q-2})$$

and

$$F(P) \equiv a\,|P|^2 + b\,|P|^q + G(P) \quad (P \in M^{n\times N})$$

for some constants $a,\ b > 0$. Then clearly (H4) is valid.

We check (H3) by fixing $x \in \mathbb{R}^n$, $r > 0$, $A \in M^{n \times N}$, $\phi \in C_0^1(B(x, r); \mathbb{R}^N)$, and calculating

$$\int_{B(x,r)} F(A + D\phi)\, dy = \int_{B(x,r)} [a\,|A + D\phi|^2 + b\,|A + D\phi|^q + G(A + D\phi)]\, dy$$
$$\geqq \int_{B(x,r)} [a\,|A|^2 + a\,|D\phi|^2 + b\,|A|^q + b\varkappa\,|D\phi|^q + G(A)]\, dy;$$

here we have used Lemma 8.2 and the quasiconvexity of G. Thus

$$\int_{B(x,r)} [F(A) + \gamma\,(1 + |D\phi|^{q-2})\,|D\phi|^2]\, dy \leqq \int_{B(x,r)} F(A + D\phi)\, dy$$

for $\gamma \equiv \min(a, \varkappa b) > 0$. □

In order to construct specific applications, we recall that a form of degree k, $1 \leqq k \leqq \min(n, N)$,

$$a(P) \equiv A^{\alpha_1 \ldots \alpha_k}_{i_1 \ldots i_k}\, p^{i_1}_{\alpha_1} p^{i_2}_{\alpha_2} \cdots p^{i_k}_{\alpha_k}, \qquad (P = ((p^\alpha_i)) \in M^{n \times N}),$$

is called *alternating* if $1 \leqq \alpha_r \leqq n$, $1 \leqq i_s \leqq N$ $(1 \leqq r, s \leqq k)$, and

$$a(P) = C \det Q,$$

where C is a constant and Q is the $k \times k$ submatrix $((p^{\alpha_r}_{i_s}))$. BALL in [4] has called a function G *polyconvex* if it can be expressed in the form

$$G(P) \equiv g(a_1(P), \ldots, a_l(P))$$

where $g: \mathbb{R}^l \to \mathbb{R}$ is convex and the $a_i(\cdot)$ are alternating forms of (perhaps different) degrees k_i, $1 \leqq k_i \leqq \min(n, N)$, $1 \leqq i \leqq l$. MORREY proved in [14] that polyconvexity implies quasiconvexity.

Accordingly it is easy to construct specific nonconvex examples satisfying the hypotheses of both the existence theory described in section 1 and the partial regularity result of Theorem 2. An important case here is the function

$$F(P) \equiv a\,|P|^2 + b\,|P|^q + \beta\,(\det P),$$

where $n = N$, $q = n(k + 2)$, $a, b > 0$, and $\beta: \mathbb{R} \to [0, \infty)$ is a convex C^2 function satisfying $\beta''(r) \leqq C(1 + |r|^k)$ for $r \in \mathbb{R}$.

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