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Regularity of minima: an invitation to the Dark Side of the Calculus of Variations

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REGULARITY OF MINIMA: AN INVITATION TO THE DARK SIDE OF THE CALCULUS OF VARIATIONS*

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Abstract. I am presenting a survey of regularity results for both minima of variational integrals, and solutions to non-linear elliptic, and sometimes parabolic, systems of partial differential equations. I will try to take the reader to the Dark Side...

Keywords: regularity, minimizers, Dark Side

MSC 2000: 49N60, 35J20, 35J70

1. Prologue: The Dark Side (came to Paseky...)

These are the "generalized" lecture notes of a course I gave at the Paseky school of Mathematical Theory in Fluid Mechanics, at the end of June 2005; "generalized" because they largely extend the presentation I offered at Paseky. The school has a great and prestigious tradition: it was founded in 1991 by Jindřich Nečas, with the help of his then young students, amongst which Eduard Feireisl, Josef Málek, Antonín Novotný, Mirko Rokyta, and Michael Růžička, which are today active organizers, as well as well-known mathematicians. Eventually, Paseky's school rapidly established its reputation as one of the leading seminars for mathematical Fluid Mechanics, and I was happy to give my contribution to the ninth edition.

A few words about the odd title of this paper, and on how such a paper finds its place in the context of a school in Fluid Mechanics. The aim of my lectures was to present a basic introduction to certain classical regularity issues, and to a few, new regularization techniques that recently emerged in order to treat some variational [5],

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[6], [66], and also non-variational [8], problems, whose non-standard structure intervene also in the setting of Non-Newtonian Fluid Mechanics [274]. Originally developed in a variational context, such methods were adapted, and extended to approach more general problems [7], [10]. Therefore I decided to present a selection of results and techniques, especially referring to the variational case, that in turn should also apply, modulo suitable re-adaptations, to non-variational situations. All this reflects the content of this paper; moreover, here I will also try to address a few open problems. *Such open problems will be often emphasized in the way the words you are reading now are.*

Why talking about a "Dark Side"? It is my—maybe wrong—impression, that today regularity problems are not as popular as they were once. Regularity methods are sometimes not very intuitive, and often overburdened by a lot of technical complications, eventually covering the main, basic ideas. The well known motto: "God (or Devil) is in the details" (of the estimates!), heavily applies here. Moreover, very often no room for partial results is given: either the whole problem is solved, or really nothing comes up! So, fewer and fewer young analysts move to face such issues, and regularity, especially in the Calculus of Variations, turns out to be in the Dark Side. This paper aims to be a "friendly invitation" to come to the Dark Side [229]. It collects some recent and non-recent regularity results for minimizers of variational integrals, and solutions to elliptic systems/equations, striving for casting a relatively general panorama in the unconstrained minimization problem case. I shall start from the by now classical stuff, mostly developed until the end of the eighties, and then I will come to some more recent material. Of course the outcome will be unavoidably partial, strongly influenced by what has been my personal research up to now, and I apologize for all that fine material which will not find its room here, together with missed quotations of important contributions. Nevertheless, I hope the reader will take up my invitation to the Dark Side, eventually also hoping for some final redemption!

2. The scalar case, and the phantom irregularity

The results presented in this section can be considered as classical, and their final settling, in most of the cases, dates back to the end of the eighties. Let me start considering variational integrals of the type

$$\mathcal{F}(v, A) := \int_A F(x, v, Dv)\, \mathrm{d}x \tag{2.1}$$

defined for Sobolev maps $v \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, and open sets A whose closure is compact and contained in Ω. Here $n \geqslant 2$, $N \geqslant 1$, Ω is a bounded open set in $\mathbb{R}^n$, $p \geqslant 1$, and

$F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ is an integrand, for simplicity assumed to be measurable with respect to the first variable, and continuous with respect to the other two. In the following I will also denote

$$\mathcal{F} \equiv \mathcal{F}(v) \equiv \mathcal{F}(v, \Omega).$$

A local minimizer of the functional $\mathcal{F}$ is a map $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ such that

$$\mathcal{F}(u, A) \leqslant \mathcal{F}(v, A),$$

whenever $A \subset\subset \Omega$ and $u - v \in W^{1,p}_0(A, \mathbb{R}^N)$. A classical problem in the Calculus of Variations consists in studying the regularity properties of such maps.

Strongly connected to this problem is the one of regularity of weak solutions to elliptic systems of the type

$$\operatorname{div} a(x, u, Du) = b(x, u, Du), \tag{2.2}$$

where $a\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ and $b\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^N$ are vector fields, again assumed to be measurable with respect to the first variable, and continuous with respect to the other two. Indeed, when the integrand $F(x, v, z)$ of $\mathcal{F}$ in (2.1) is regular enough, minimizers of $\mathcal{F}$ solve the so called Euler-Lagrange system associated to $\mathcal{F}$

$$\operatorname{div} F_z(x, u, Du) = F_v(x, u, Du), \tag{2.3}$$

which turns out to be elliptic provided $F(x, v, z)$ satisfies suitable convexity assumptions with respect to z, see [152], Chapters 1 and 2. The symbol F_z denotes of course the partial derivative of F with respect to the gradient variable z. A good reference for regularity results for elliptic systems is also [31].

Throughout the paper I will present a list of theorems and results, almost never under the most general assumptions; I will rather prefer to confine myself to the simplest, basic cases, in order to emphasize the main ideas. The interested reader will find more material, and results under optimal assumptions, in the references that I am going to provide.

In this paper, I shall usually adopt the following viewpoint: given a local minimizer $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ of the functional $\mathcal{F}$, or a weak solution to (2.2), what are the additional regularity properties of u, in the interior of Ω? So, I will not discuss, for instance, the regularity of u up to the boundary; that is also why no assumptions are made on the smoothness of $\partial\Omega$ in what follows. More importantly, I will not address existence problems: for these the reader may look at, for example, the two books by

Dacorogna [72] and Giusti [165], as far as minimizers are concerned; for equations and systems of the type (2.2), the well-known monotone operators theory generally applies [228]. Giusti's book is also a very good and smooth introduction to some of the regularity topics I am going to deal with in the present review.

In this section I shall focus on the scalar case $N = 1$, later on I shall deal with the *vectorial one*, that is $N > 1$ (sometimes I will refer to the vectorial case indicating $N \geqslant 1$, when I will present results valid for both the cases). In the scalar case both (2.2) and (2.3) become nonlinear elliptic equations in divergence form provided that F is smooth enough to ensure that the terms in (2.1) are meaningful. A classical reference for these is of course [213]. We shall see that in the scalar case, under suitable assumptions on the integrand $F(x, v, z)$ in (2.1) and the vector field $a(x, v, z)$ in (2.2), it is possible to build a satisfying regularity theory, and irregularity of minima and solutions remains a phantom menace.

Let me fix an important notation here: throughout the paper ν, L and p will denote three real numbers such that

$$0 < \nu \leqslant L < \infty, \quad p > 1.$$

2.1. Hölder regularity

I shall start by the "following important fundamental result" of De Giorgi [77], so defined in Morrey's review [252]. Let me consider a linear elliptic equation in divergence form, with bounded and measurable coefficients: $\operatorname{div}(a_{i,j}(x)D_j u) = 0$, that is, in its weak formulation,

$$\int_\Omega a_{i,j}(x) D_j u D_i \varphi \,\mathrm{d}x = 0, \quad \forall\, \varphi \in C_c^\infty(\Omega). \tag{2.4}$$

The equation has bounded and elliptic coefficients $\{a_{i,j}(x)\}$, which are nevertheless supposed to be only measurable:

$$|a_{i,j}(x)| \leqslant L, \quad a_{i,j}(x)\lambda_i\lambda_j \geqslant \nu|\lambda|^2, \tag{2.5}$$

for a.e. every $x \in \Omega$ and every $\lambda \in \mathbb{R}^n$. In the sequel I will go on using the usual summation convention on repeated indexes. It is clear that the role of ν is that of a lower bound for the eigenvalues of the matrix $\{a_{i,j}(x)\}$, while L acts as an upper one. We have

Theorem 2.1 (De Giorgi). *Let $u \in W^{1,2}(\Omega)$ be a weak solution to the equation* (2.4) *under the assumptions* (2.5). *Then there exists a positive number $\alpha \equiv \alpha(n, L/\nu) > 0$ such that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$.*

This result was independently obtained by Nash [264], directly for parabolic equations. Anyway, Nash's techniques have not led to massive developments of De Giorgi's: see the comments in the introduction of [215], a book where De Giorgi's methods are extended to the parabolic case in great extent; but see also the paper [117], where Nash's techniques are revitalized. A little later Moser [258], [259], [260] gave different proofs of De Giorgi's and Nash's results, proving actually the validity of Harnack's inequality for solutions to (2.4), and to its parabolic analog

$$\int_{\Omega\times[0,T)} u\varphi_t - a_{i,j}(x,t)D_j u D_i \varphi \,\mathrm{d}x\,\mathrm{d}t = 0, \quad \forall\, \varphi \in C^\infty_c(\Omega \times [0,T)).$$

Indeed, Harnack's inequality implies in turn local Hölder continuity of solutions, see [165], notes to Chapter 7. Nowadays such basic regularity methods are indeed known as De Giorgi-Nash-Moser's theory. For an original and elegant approach to such theory see also [31], Chapter 2.

In Theorem 2.1 the dependence of the Hölder continuity exponent α is critical with respect to the "ellipticity ratio" L/ν of the matrix $\{a_{i,j}(x)\}$:

$$\lim_{L/\nu\to\infty} \alpha = 0. \tag{2.6}$$

Indeed, the main strength of Theorem 2.1 is in the fact that the coefficients $\{a_{i,j}(x)\}$ are allowed to be merely measurable; in the case of continuous coefficients, the result had been in fact already known, and proved via perturbation methods, i.e. the so called Korn's trick; see also Schauder estimates techniques in Chapter 6 of [164]. In this case the solution turns out to be actually locally Hölder continuous with any exponent $\alpha < 1$, an effect of the continuity of coefficients. The difference of Theorem 2.1 from the continuous coefficients case is emphasized by (2.6), which reflects the basic role of the sole ellipticity and growth assumptions (2.5). The importance of De Giorgi's theorem and technique is manifold: De Giorgi was initially motivated to prove it, apparently after discussions with Stampacchia, with the aim of solving the famous Hilbert's 19th problem; I refer to the survey of Marcellini [240] for an updated discussion of it, and to the older survey by Stampacchia [286]. Even more importantly, as we shall see in a few lines, De Giorgi's insights opened the way to the nonlinear theory, and they are the cornerstone of what is nowadays called "Non-linear Potential Theory", see [181], [230], [227], [300].

De Giorgi's proof rested on a then completely new method. Roughly speaking, it is based on the idea of proving regularity properties of solutions via the analysis of the decay and density properties of their level sets, a method that eventually became pervasive in the whole regularity theory. Indeed, De Giorgi's proof starts with the observation that a weak solution to (2.4) satisfies the following "Caccioppoli type inequalities" [50] on level sets:

$$
\begin{cases} \displaystyle\int_{A(k,\varrho)} |Du|^2 \,\mathrm{d}x \leqslant \frac{c}{(R-\varrho)^2} \int_{A(k,R)} |(u-k)_+|^2 \,\mathrm{d}x, \\ \displaystyle\int_{B(k,\varrho)} |Du|^2 \,\mathrm{d}x \leqslant \frac{c}{(R-\varrho)^2} \int_{B(k,R)} |(k-u)_-|^2 \,\mathrm{d}x. \end{cases} \tag{2.7}
$$

Here $0 < \varrho < R$, and with $s > 0$

$$
A(k,s) := \{x \in B_s\colon\ u(x) \geqslant k\}, \quad B(k,s) := \{x \in B_s\colon\ u(x) \leqslant k\},
$$

with $B_s \subset\subset \Omega$ denoting a ball of radius s, while $(u-k)_+ := \max\{u-k, 0\}$ and $(k-u)_- := \max\{k-u, 0\}$. The constant c depends essentially on the ellipticity ratio L/ν. From this only information, via an innovative iteration procedure, De Giorgi was able to derive the Hölder continuity of solutions, with an exponent depending on c, and therefore ultimately on L/ν. So, the whole Hölder continuity information of solutions is encoded in the the two inequalities (2.7); this motivates the nowadays common definition stating that a function u is in De Giorgi's class DG iff it satisfies (2.7) for all possible choices of k, ϱ and R. Extensions, and a gentle introduction to De Giorgi's method, can be found in [165], Chapter 7, and [230], Chapter 2. See also the original papers [253], [285], [279], [296], [297], [299], and again the monograph [213], where the original De Giorgi's and Moser's methods have been deeply extended and clarified.

It was soon recognized that the linearity of the equation (2.4) played actually no role in the proof of (2.7), the ideas involved being genuinely non-linear ones, and the result was rapidly extended to a vast class of general nonlinear elliptic equations in divergence form [213]. The following result is an example. Let me consider an elliptic equation of the type

$$
\operatorname{div} a(x, u, Du) = 0, \tag{2.8}
$$

under the following growth and monotonicity assumptions:

$$
|a(x,v,z)| \leqslant L(1+|z|^{p-1}), \quad \nu|z|^p - L \leqslant \langle a(x,v,z), z\rangle \tag{2.9}
$$

for every $x \in \Omega$, $v \in \mathbb{R}$ and $z \in \mathbb{R}^n$, with $p > 1$. Then we have

Theorem 2.2. *Let $u \in W^{1,p}(\Omega)$ be a weak solution to the equation* (2.8) *under the assumptions* (2.9). *Then there exists a positive number $\alpha \equiv \alpha(n, p, L/\nu) > 0$ such that $u \in C^{0,\alpha}_{\rm loc}(\Omega)$.*

Once again the proof, see for instance [230], rests on proving that a solution satisfies inequalities similar to the ones in (2.7), with the growth exponent p replacing 2, and then on applying De Giorgi's method; the dependence of α is the same as the one in (2.6). No pointwise regularity property of the vector field a is required with respect to the variables (x, v). Extensions are possible to complete equations of the type (2.2), including lower order terms in the formulation of the assumptions (2.9), see [213] or [230]. For the extension of such result to parabolic equations of the form

$$u_t - \operatorname{div} a(x, u, Du) = 0 \tag{2.10}$$

I refer to [215] and, especially for the degenerate case including the evolutionary p-Laplacean equation $u_t - \operatorname{div}(|Du|^{p-2}Du) = 0$, to DiBenedetto's book [81]. Such an extension to degenerate problems is highly non-trivial, and involves DiBenedetto's innovative method of intrinsic geometry: using, in the formulation of the (parabolic) Caccioppoli's estimates, a parabolic cylinder whose size is determined by the solution itself.

Of course it was immediately observed that, for functionals of the type (2.1) whose associated Euler-Lagrange equation (2.3) satisfies assumptions of the type (2.9), the Hölder regularity of minimizers follows if they are viewed as solutions to equations of the type (2.2). But it took not less than twenty years to start exploiting the full impact of De Giorgi's techniques on the regularity of minima. Indeed, first Frehse [132], under stronger assumptions, and then Giaquinta & Giusti [157], in full generality, applied De Giorgi's method to minimizers in a direct way, that is without using the Euler-Lagrange equation, which may possibly not exist. More precisely, considering only the following growth assumptions on the integrand $F(x, v, z)$:

$$\nu|z|^p \leqslant F(x, y, z) \leqslant L(1 + |z|^p), \tag{2.11}$$

we have

Theorem 2.3. *Let $u \in W^{1,p}(\Omega)$ be a local minimizer of the functional $\mathcal{F}$ under the assumptions* (2.9). *Then there exists a positive number $\alpha \equiv \alpha(n, p, L/\nu) > 0$ such that $u \in C^{0,\alpha}_{\rm loc}(\Omega)$.*

The proof in [157] is elegant and simple, and makes use of a clever application of the hole-filling technique of Widman [304]. It essentially relies on the observation that the sole minimality property, and growth conditions (2.11), force minimizers to satisfy inequalities of the type in (2.7), with p replacing the exponent 2; then Hölder continuity automatically follows via De Giorgi's iteration method. Therefore the result is valid for functions $F(x, v, z)$ which are not differentiable with respect to the v-variable, and even not convex with respect to the gradient variable z; moreover, the result extends to the so-called ω-minima, see Subsection 4.5 below. Shortly later, DiBenedetto & Trudinger [84] proved that minimizers also satisfy the Harnack inequality under the assumptions of Theorem 2.3, thus extending to very irregular functionals Moser's results for elliptic equations. Once again, DiBenedetto & Trudinger directly proved the validity of Harnack's inequality not only for minima, but more generally for functions in De Giorgi's classes DG. *A full extension of such Harnack inequalities results to the general parabolic case* (2.10) *is nevertheless still an open problem, see* [80].

Added in proof. In an extremely recent paper, ("Harnack estimates for quasi-linear degenerate parabolic differential equations", 2006) Di Benedetto & Gianazza & Vespri proved the validity of suitable Harnack inequalities for possibly degenerate parabolic equations of the type (2.10), when $p \geqslant 2$. More precisely, considering equation (2.10) in the cylindrical domain $\Omega \times (0, T]$, and the intrinsic parabolic space-time cylinders "centered" at (x_0, t_0) of the type

$$(x_0, t_0) + B_R(0) \times (-\theta R^p, 0] \subset \Omega \times [0, T),$$

then either

$$u(x_0, t_0) \leqslant C_1 R,$$

or

$$u(x_0, t_0) \leqslant C_1 \inf_{B_R(x_0)} u(x, t_0 + \theta R^p), \tag{2.12}$$

where the crucial point is that θ depends on the solution itself:

$$\theta := \Big(\frac{c}{u(x_0, t_0)}\Big)^{p-2}.$$

The constant C_1 only depends on n, ν, L, p. Also, the previous result reduces the classical one valid for the heat equation when $p = 2$, when $\theta = 1$. Note that the size of the cylinder considered depends on the solutions itself, accordingly to the typical behavior of such parabolic equations identified by DiBenedetto [63]. This means the following: when looking at the model equation

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = 0,$$

we see that it is not invariant under re-scaling, i.e.: *multiplying a solution times a constant does yield another solution*, unless $p = 2$. For this reason there is no natural family of cylinders associated with parabolic equations of the type in (2.10), and the cylinders to be considered to analyze the regularity of solutions are defined in an intrinsic way, as for instance, those ones coming into play when proving Lipschitz estimates:

$$K := (x_0, t_0) + B_R(0) \times (-\lambda^{2-p} R^2, 0]. \tag{2.13}$$

Here $\lambda > 0$ is such that (this is very rough, and not completely precise)

$$\lambda \approx \left(\fint_K |Du|^p \, \mathrm{d}x \, \mathrm{d}t \right)^{1/p}.$$

Note that the critical point is that λ appears on both the sides of the previous equivalence via the definition of K, and it is part of the proof to show that such intrinsic cylinders can be actually considered: this is the core of DiBenedetto's approach to parabolic regularity. For the same reason, in order to obtain regularity estimates on small subsets, it is not possible to use scaling arguments, as in the elliptic case, unless again $p = 2$. Observe that when $p = 2$ we have a natural, general family of cylinders associated with the problem, the well-known parabolic ones:

$$(x_0, t_0) + B_R(0) \times (-R^2, 0],$$

that is the ones in K when $p = 2$, and therefore scaling arguments work. The peculiar form of the Harnack type inequality in (2.12) is just another instance of such general facts. Anyway, *the problem of proving intrinsic Harnack inequalities for solutions to general equations as* (2.10) *still remains open in the sub-quadratic case* $2 > p > 2n/(n + 2)$.

Coming back to the elliptic case, let me observe that De Giorgi's techniques open the way to study low order regularity also for equations and functionals with coefficients in Lorenz spaces, see for instance [121].

2.2. Lipschitz type regularity

Up to now we have seen what are the assumptions implying local Hölder continuity of minima and solutions to equations *for some Hölder exponent* $\alpha > 0$. Now I will review some higher regularity results; not surprisingly, in order to have higher regularity of solutions and minimizers, one must assume more regularity on the vector field a in (2.8), and on the integrand $F(x, v, z)$ in (2.1), as shown by the examples in [269], [221].

I shall start presenting an innovative result by Fonseca & Fusco [122], [123], [110], concerning integral functionals of the type

$$\mathcal{G}_s(v) := \int_\Omega \nu|Dv|^p + g(Dv)\,\mathrm{d}x. \tag{2.14}$$

Theorem 2.4. *Let $u \in W^{1,p}(\Omega)$ be a local minimizer of the functional $\mathcal{G}_s$, such that $g\colon \mathbb{R}^n \to \mathbb{R}^+$ is a convex function satisfying $0 \leqslant g(z) \leqslant L(1+|z|^p)$ for some $L \geqslant 0$. Then $Du \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$.*

The significance of this result lies in the fact that the regularity assertion is this time made on the gradient Du, and this is usually achieved by using the Euler-Lagrange equation of the functional, which has in turn to be differentiated again. This allows to discover that the gradient itself is a solution to another equation, and then to argue on this. In the case of the functional $\mathcal{G}_s$, the integrand has the form $F(z) = \nu|z|^p + g(z)$, and the general assumptions on g only allow to conclude that F is differentiable once, and only at almost every point, being anyway a convex and therefore Lipschitz function. Fonseca & Fusco by-passed this point by combining essentially two ingredients: a delicate way of deriving a priori L^∞-estimates for the gradient Du when dealing with more regular integrands, and a suitable approximation argument in order to approximate the functional $\mathcal{G}_s$ with a sequence of smoother ones, satisfying the Euler-Lagrange equation. Observe that in Theorem 2.4 the number $\nu > 0$ can be picked small at will without any loss of regularity on Du. An extension to Theorem 2.4 to more general functionals of the type

$$\mathcal{G}_s(v) := \int_\Omega \nu|Dv|^p + g(x, v, Dv)\,\mathrm{d}x \tag{2.15}$$

is possible, this time requiring in addition that g is continuous with respect to the variable (x, v), uniformly with respect to z, that is

$$|g(x,u,z) - g(y,v,z)| \leqslant L\omega(|x-y| + |u-v|)(1+|z|^p)$$

for every $x, y \in \Omega$, $u, v \in \mathbb{R}$ and $z \in \mathbb{R}^n$. Here $\omega\colon [0,\infty) \to [0,1]$ is a non-decreasing continuous function vanishing at zero, what from now on I shall call a "modulus of continuity". In this case we have that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ *for every* $\alpha < 1$. This result has been achieved in [68], and requires a further refinement of the techniques in [122], based on the so called Ekeland's variational principle [105]. The use of this last tool to attack regularity problems was first introduced, in a different context, by Fusco & Hutchinson in [142]; it eventually became a standard. It is to note that now α is independent of the ratio L/ν, unlike in (2.6); this is the combined effect of the global strict convexity with respect to z exhibited by $F(x,v,z)=\nu|z|^p+g(x,v,z)$, and the fact that the dependence on (x,v) is not just measurable, but now, rather, continuous. Further extensions are in [70].

2.3. $C^{1,\alpha}$-regularity

I will finally pass to examine situations where higher regularity can be achieved. I will confine myself to consider local Hölder continuity of the gradient of solutions and minima. In fact, this is the focal point of the theory. Once this type of regularity is achieved, then higher regularity of solutions, up to analyticity, can be obtained by well-known boot-strap methods; I shall not dedicate space to this ultra-classical point, which is essentially based on the so called Schauder estimates for linear elliptic systems and equations with variable coefficients; the reader is referred for instance to [165], Chapter 10, for a neat and elementary presentation.

The first result I am going to report on concerns integral functionals of the type $\mathcal{F}$. The assumptions will be this time

$$
\begin{cases}
z\mapsto F(x,v,z) \ \text{ is } \ C^2,\\
\nu|z|^p\leqslant F(x,v,z)\leqslant L(1+|z|^p),\\
\nu(\mu^2+|z|^2)^{(p-2)/2}|\lambda|^2\leqslant\langle F_{zz}(x,v,z)\lambda,\lambda\rangle\leqslant L(\mu^2+|z|^2)^{(p-2)/2}|\lambda|^2,\\
|F(x,u,z)-F(y,v,z)|\leqslant L\omega(|x-y|+|u-v|)(1+|z|^p)
\end{cases}
\tag{2.16}
$$

for all $x,y\in\Omega$, $u,v\in\mathbb{R}$ and $z,\lambda\in\mathbb{R}^n$, where $\mu\in[0,1]$ is a fixed constant and $\omega\colon\ \mathbb{R}^+\to(0,1)$ is a continuous, non-decreasing modulus of continuity, such that for some $\alpha\in(0,1)$,

$$
\omega(s)\leqslant s^{\alpha}.
\tag{2.17}
$$

Assumption $(2.16)_3$ describes a controlled uniform convexity of the integrand F, via growth conditions imposed on the second derivatives F_{zz}, which are once again prescribed accordingly to the ones in $(2.16)_2$. On the other hand, assumption $(2.16)_4$, together with (2.17), means that the integrand F is Hölder continuous with respect to (x,v) with an exponent $\alpha\in(0,1)$, uniformly with respect to z. Note that the Hölder continuity condition has been re-normalized taking into account the growth conditions in $(2.16)_2$. Roughly speaking, when prescribing (2.16), one thinks of model examples such as $F(x,v,z)\equiv c(x,v)(1+|z|^2)^{p/2}$ or $F(x,v,z)\equiv c(x,v)f(z)$, and $\nu\leqslant c(x,v)\leqslant L$.

In the previous assumptions the parameter μ plays a very important role. When $\mu>0$, the functional is *non-degenerate elliptic*. The case $\mu=0$ corresponds to degenerate cases. For instance, a model case when $\mu>0$ is given by

$$
\int_\Omega c(x,v)(1+|Dv|^2)^{p/2}\,\mathrm{d}x
$$

where $\nu \leqslant c(x,v) \leqslant L$ is a Hölder continuous function; here of course $\mu = 1$. A typical degenerate model is

$$\int_\Omega c(x,v)|Dv|^p \, dx.$$

In the case $c(x,v) \equiv 1$ we have the p-Dirichlet functional

$$\mathcal{D}_p(v) := \int_\Omega |Dv|^p \, dx, \tag{2.18}$$

where of course we are taking $\mu = 0$, and whose Euler-Lagrange equation is the following well-known, degenerate p-Laplacean equation:

$$\operatorname{div}(|Du|^{p-2}Du) = 0. \tag{2.19}$$

Actually in (2.16) the only important cases are the degenerate one $\mu = 0$, and $\mu = 1$, the case we can always reduce to when $\mu > 0$, provided we increase the ratio L/ν enough, depending on how μ is close to 0.

I shall present two theorems; the first concerns the non-degenerate case $\mu > 0$:

Theorem 2.5. *Let $u \in W^{1,p}(\Omega)$ be a local minimizer of the functional $\mathcal{F}$ under the assumptions* (2.16), *with $\mu > 0$. Then $Du \in C^{0,\alpha/2}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$.*

It is to be noted that the degree of regularity of $F(x,v,z)$ with respect to (x,v) directly influences the degree of regularity of the gradient. This is a well known phenomenon, think of Schauder estimates for linear elliptic equations [164]. The second result regards the degenerate case $\mu = 0$.

Theorem 2.6. *Let $u \in W^{1,p}(\Omega)$ be a local minimizer of the functional $\mathcal{F}$ under the assumptions* (2.16) *with $\mu = 0$. Then $Du \in C^{0,\beta}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ for some $\beta \equiv \beta(n,p,L,\alpha) > 0$.*

In this last case there is a loss in the Hölder continuity degree of the gradient, due to the fact that the problem is actually degenerate. This means the following, looking at equation (2.19): letting $a(z) := |z|^{p-2}z$, the lowest eigenvalue of the matrix a_z is comparable to $|z|^{p-2}$. So, when $|Du|$ approaches zero, the equation (2.19) loses its ellipticity properties and estimates worsen. Indeed, even in the case of solutions to (2.19) and minima of (2.18), the gradient Du is not β-Hölder continuous with any exponent $\beta < 1$, as shown by Ural'tseva in 1968 [302]. On the contrary, minima of

$$\int_\Omega (\mu^2 + |Dv|^2)^{p/2} \, dx, \quad \mu > 0,$$

are $C^\infty(\Omega)$: in this case the associated Euler-Lagrange equation is non-degenerate elliptic. Theorem 2.6, and the present form of Theorem 2.5, are due to Manfredi [231], a paper I refer to for further references on degenerate problems; the first form of Theorem 2.5, under more restrictive assumptions, and in particular for the case $p \geqslant 2$, and in the non-degenerate case $\mu > 0$, is independently due to Giaquinta & Giusti [158], [159], and Ivert [188]. Once again, Theorems 2.5 and 2.6 are significant because the regularity of the gradient is obtained for functionals that do not necessarily satisfy the Euler-Lagrange equation (2.3); the functionals considered are indeed non-differentiable since the dependence on the variable v of the integrand F is just Hölder continuous, and therefore F_v does not exist in general. This was the main contribution in [188], [158].

Theorems 2.5 and 2.6 have their counterparts for elliptic equations. For simplicity I shall report on the homogeneous case (2.8). Let me consider the following assumptions, which are the natural reformulation of the ones in (2.16) when arguing on the vector field a, rather than on the integrand F:

$$\begin{cases} z \mapsto a(x,v,z) \text{ is } C^1, \\ |a(x,v,z)| \leqslant L(1+|z|^{p-1}), \\ \nu(\mu^2+|z|^2)^{(p-2)/2}|\lambda|^2 \leqslant \langle a_z(x,v,z)\lambda,\lambda\rangle \leqslant L(\mu^2+|z|^2)^{(p-2)/2}|\lambda|^2, \\ |a(x,u,z)-a(y,v,z)| \leqslant L\omega(|x-y|+|u-v|)(1+|z|^{p-1}) \end{cases} \tag{2.20}$$

for all $x,y \in \Omega$, $u,v \in \mathbb{R}$ and $z,\lambda \in \mathbb{R}^n$, where $\mu \in [0,1]$ is a fixed constant and $\omega\colon \mathbb{R}^+ \to (0,1)$ is as in (2.17). Then we have the natural analogs of Theorems 2.5 and 2.6:

Theorem 2.7. *Let $u \in W^{1,p}(\Omega)$ be a weak solution to the equation* (2.8) *under the assumptions* (2.20), *with $\mu > 0$. Then $Du \in C^{0,\alpha}_{\text{loc}}(\Omega,\mathbb{R}^n)$.*

As for Theorem 2.7 compared to Theorem 2.5, let me notice that the degree of Hölder continuity of the gradient increases, from $\alpha/2$ to α. This is a general principle: minimality itself is a property strong enough in order to guarantee regularity also when dealing with irregular, non-differentiable functionals, but the property of satisfying an equation is stronger and forces higher regularity. The degenerate analog is finally the following:

Theorem 2.8. *Let $u \in W^{1,p}(\Omega)$ be a weak solution to the equation* (2.8) *under the assumptions* (2.20) *with $\mu = 0$. Then $Du \in C^{0,\beta}_{\text{loc}}(\Omega,\mathbb{R}^n)$ for some $\beta \equiv \beta(n,p,L,\alpha) > 0$.*

The proof of Theorems 2.5–2.8 is based on a comparison argument using the so called freezing method. In their full, "degenerate" generality, Theorems 2.7 and 2.8

are due to Manfredi [231]. Unfortunately, many of the interesting estimates Manfredi developed are not included in the paper [231], but are nevertheless retrievable in his Ph.D. thesis [232], which for its clearness of exposition I highly recommend as a first approach to the subject. For degenerate elliptic problems in the vectorial case, and for further techniques, see [143], [171].

3. The revenge of irregularity: the vectorial case

There were still hopes for getting a vectorial version of De Giorgi's Theorem 2.1 around 1967, when De Giorgi himself showed that no such extension could take place.

3.1. De Giorgi's example [79]

This actually deals with functionals, and therefore also simultaneously shows that no extension to Theorems 2.1 and 2.3 can be achieved when $N > 1$. The counterexample for systems follows of course by considering the corresponding Euler-Lagrange system. De Giorgi considered the following quadratic-type functional with discontinuous coefficients:

$$\mathcal{DG} := \int_{B_1} |Du|^2 + \left[(n-2)\sum_{i=1}^{n} D_i u^i + n \sum_{i,j=1}^{n} \frac{x_i x_j}{|x|^2} D_i u^j\right]^2 \mathrm{d}x, \quad n = N. \tag{3.1}$$

For $n \geqslant 3$ the map

$$u(x) := \frac{x}{|x|^{\alpha}}, \quad \alpha := \frac{n}{2}\left[1 - \frac{1}{\sqrt{(2n-2)^2+1}}\right], \tag{3.2}$$

belongs to $W^{1,2}(B_1, \mathbb{R}^n)$, and locally minimizes $\mathcal{DG}$. Let me just observe that in this wonderfully simple example it is necessary to take discontinuous dependence on x in the integrand, otherwise the solutions are Hölder continuous, and with any exponent $\alpha < 1$, to use perturbation methods. No such example is possible when $n = 2$, since in this case Hölder continuity of solutions, for some exponent $\alpha < 1$, follows by the higher integrability guaranteed by Theorem 4.1 below, and by Sobolev's embedding theorem. Let me point out that almost independently and at the same time, Maz'ya [247] found an example of higher order elliptic equations, with analytic coefficients, but with discontinuous solutions.

3.2. Giusti & Miranda's example [166]

The main point in De Giorgi's example is the singularity of the matrix $\{a^{\alpha,\beta}_{i,j}(x)\}$ at the origin. When the coefficients matrix depends on the solution Giusti & Miranda showed that the matrix $\{a^{\alpha,\beta}_{i,j}(v)\}$ can be even analytic. They considered the quadratic-type functional

$$\mathcal{GM}(v) := \int_{B_1} a_{i,j}^{\alpha,\beta}(v) D_j v^\alpha D_i v^\beta \, \mathrm{d}x, \quad n = N, \tag{3.3}$$

with

$$a_{i,j}^{\alpha,\beta}(v) := \delta_{i,j}\delta_{\alpha,\beta} + \left[\delta_{\alpha,i} + \frac{4}{n-2} \cdot \frac{v_\alpha v_i}{1+|v|^2}\right] \cdot \left[\delta_{\beta,j} + \frac{4}{n-2} \cdot \frac{v_\beta v_j}{1+|v|^2}\right].$$

Here $\delta_{i,j}$ denotes the usual Kronecker's symbol. For $n > 2$ sufficiently large, the discontinuous map

$$u(x) := \frac{x}{|x|} \tag{3.4}$$

locally minimizes $\mathcal{GM}$. Similar examples also work for quasilinear systems of the type

$$\int_\Omega a_{i,j}^{\alpha,\beta}(u) D_j u^\alpha D_i \varphi^\beta \, \mathrm{d}x = 0, \quad \forall \, \varphi \in C_c^\infty(\Omega, \mathbb{R}^n). \tag{3.5}$$

The key to understand why in this example analytic coefficients are allowed, in contrast to the one in Subsection 3.1, is that if we think of the identification $b_{i,j}^{\alpha,\beta}(x) \equiv a_{i,j}^{\alpha,\beta}(u(x))$, then the map $x \to b_{i,j}^{\alpha,\beta}(x)$ is just measurable. Remarkable extensions of De Giorgi's and Giusti & Miranda's constructions are given in [284], [194], where nowhere continuous solutions to elliptic systems are produced via the construction of a suitable, very pathological coefficient matrix $\{a_{i,j}^{\alpha,\beta}\}$.

3.3. Nečas' example [265], [176]

In the previous examples the singularity of minimizers occurs due to the peculiar way the discontinuous coefficients (x, v) mix up with the components of the gradient variable z. What happens when there are no coefficients? This question was first answered by Nečas, who considered a simple functional of the type

$$\mathcal{F}_s(v) := \int_\Omega F(Dv) \, \mathrm{d}x, \tag{3.6}$$

and whose example immediately applies to systems when considering the Euler-Lagrange system associated to $\mathcal{F}_s$,

$$\operatorname{div} F_z(Du) = 0. \tag{3.7}$$

In Nečas' example the integrand $F : \mathbb{R}^{nN} \to \mathbb{R}^+$ is analytic, with quadratic growth, and satisfies the uniform ellipticity and growth conditions

$$\nu|\lambda|^2 \leqslant \langle F_{zz}(z)\lambda, \lambda\rangle \leqslant L|\lambda|^2 \tag{3.8}$$

for all $z, \lambda \in \mathbb{R}^{nN}$. The integrand $F(z)$ is rather complicated, and it can be found in [176], formula (3.1). With $\Omega \equiv B_1 \subset \mathbb{R}^N$, the minimizer considered this time is the map $u\colon B_1 \to \mathbb{R}^{n^2}$ defined by

$$u^{i,j}(x) := \frac{x_i x_j}{|x|} - \frac{1}{n}\delta_{i,j}|x|. \tag{3.9}$$

The importance of this example also lies in the fact that it shows that the irregularity of minima is a peculiar feature of the vectorial case, and is not due to the presence of coefficients. Nečas' example only works for $n \geqslant 5$, while in the case of (non-minimizing) solutions to systems there exists an example starting with $n \geqslant 3$ [266]. Note that the map in (3.9) is not C^1 but still Lipschitz continuous, therefore the example relates to Theorem 2.5, but not to Theorem 2.3. The problem of finding non-Lipschitz minimizers remained an important open issue in the theory for more than 25 years...

3.4. Šverák & Yan's example [290], [291]

...until it was settled by Šverák & Yan, who showed that minimizers of analytic functionals as considered by Nečas, and therefore also satisfying (3.8), may be even unbounded! The construction offerred in [291] does not produce an explicit formula for the integrand $F(z)$; the minimizer $u\colon B_1 \to \mathbb{R}^{n^2}$ considered this time is a variant of Nečas' in the sense that

$$u^{i,j}(x) := \frac{1}{|x|^\varepsilon}\Big(\frac{x_i x_j}{|x|} - \frac{1}{n}\delta_{i,j}|x|\Big)$$

for suitable positive values of ε. Moreover, Šverák & Yan also offer a completely new example of a non-Lipschitz minimizer in low dimensions, taking $n = 4$ and $N = 3$, and also give an example of a non-Lipschitz minimizer when $n = 3$ and $N = 5$. *It still remains an open problem to find a non-C^1 minimizer for the case $n = N = 3$*, which plays a role in the applications. Note anyway that, by higher differentiability and integrability of solutions and Sobolev's embedding theorem, minima are Hölder continuous up to dimension $n = 4$, and in this sense Šverák & Yan's example of unbounded minimizers is dimensionally optimal; see the first section of [291] for more comments.

4. A new hope: partial regularity

We have just seen that in the general vectorial case $N > 1$ both the minima of variational integrals and the solutions to elliptic systems may develop singularities. Moreover, at least by looking at the phenomena observed up to now, we may say that everywhere regularity occurs very rarely in the vectorial case. On the other hand, especially in geometrically constrained problems, it is possible to see that *minimizing the energy naturally creates singularities.* For instance, taking $\Omega \equiv B_1 \subset \mathbb{R}^n$, the map $x \mapsto x/|x|$, already met in (3.4), minimizes the functional $\mathcal{D}_p$ in (2.18) in its Dirichlet class when $p < n$, among all maps taking values in the unit sphere of $\mathbb{R}^n$. This happens for very topological reasons. See the most recent contributions [178], [186] and related references. For such issues I recommend to have a look at [177].

4.1. Higher integrability

Anyway, a few weaker forms of regularity still persist. The only global regularity property, surviving in general the passage from the scalar to the vectorial case, is the so called "higher integrability", and it is the content of the following

Theorem 4.1. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$ under the assumption* (2.11). *Then there exists a positive number $s(n, p, L/\nu) > p$ such that $Du \in L^s_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN})$.*

This theorem has been obtained first by Attouch & Sbordone in a particular situation [22], and then by Giaquinta & Giusti [157] in the full generality considered here. In the previous statement, as in the rest of the section, $z \in \mathbb{R}^{nN}$, and whenever considered, all the assumptions stated in Section 2 must be recast taking into account this fact, and that now $u\colon \Omega \to \mathbb{R}^N$.

The proof of the previous theorem is based again on Caccioppoli's inequality in the form

$$\int_{B_{R/2}} |Du|^p \,\mathrm{d}x \leqslant \frac{c}{R^p} \fint_{B_R} |u - u_R|^p \,\mathrm{d}x + cR^n$$

for the ball $B_R \subset\subset \Omega$, and u_R is the average of u over B_R; here $c \equiv c(L/\nu)$. Then one applies Poincaré inequality to get

$$\fint_{B_{R/2}} |Du|^p \,\mathrm{d}x \leqslant \left(\fint_{B_R} |Du|^{np/(n+p)} + 1 \,\mathrm{d}x \right)^{(n+p)/n},$$

obtaining what is usually called a reverse Hölder inequality with increasing support. At this point one uses Gehring's lemma in one of its local versions [288], [161], [44], and concludes the existence of a *higher integrability exponent* $s \equiv s(L/\nu) > p$ such that

$$\left(\fint_{B_{R/2}} |Du|^s \,\mathrm{d}x\right)^{1/s} \leqslant \left(\fint_{B_R} |Du|^p + 1 \,\mathrm{d}x\right)^{1/p}.$$

In other words, when passing to the vectorial case, Caccioppoli inequalities are still able to provide some regularity, in the form of higher integrability. Needless to say, the same applies to systems:

Theorem 4.2. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a weak solution to the system* (2.8) *under the assumptions* (2.9). *Then there exists a positive number $s(n, p, L/\nu) > p$ such that $Du \in L^s_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN})$.*

Concerning the exponent s from Theorems 4.1 and 4.2, this is *explicitly computable*, in the sense that lower estimates for it are available using the ones from Gehring's lemma [44], [197]. Anyway, sharp bounds in terms of the ellipticity ratio L/ν are not known except for two dimensional, linear elliptic equations [221], when they rest on a deep theorem of Astala [21], see also [193]. A very interesting extension of Theorem 4.2 to the case of parabolic systems with p-growth has been achieved by Kinnunen & Lewis in [198]; for the case $p = 2$ see [163].

Reverse Hölder inequalities and higher integrability results were first obtained by Gehring for quasi-conformal mappings in his epoch-making paper [150], while the application to higher order equations and systems was obtained by Elcrat & Meyers [106]. Local extensions, suitable for further applications to regularity problems, were obtained by Giaquinta & Modica [161], and Stredulinsky [288]. I also recommend to have a look at the very nice proof given by Bojarski & T. Iwaniec [44]; extensions to the setting of Orlicz spaces, and in certain limit function spaces, are also available [146], [148], [191], [192], [120], [41]. For further properties and information concerning reverse Hölder inequalities and higher integrability, I again recommend the nice surveys by T. Iwaniec [191], and Sbordone [277], and the thesis of Kinnunen [197], where a detailed study of the various constants occurring in Reverse Hölder inequalities is cleverly carried out. Related earlier results are in the works of Bojarski & Sbordone & Wik [45], and D'Apuzzo & Sbordone [75]. For connections to Harmonic Analysis I recommend the paper [67] and its references.

4.2. Partial $C^{1,\alpha}$-regularity

Concerning the pointwise regularity (in the interior of Ω) of minima and solutions, the so called partial regularity comes into the play. The general principle of partial regularity asserts *the pointwise regularity of solutions/minimizers, in an open subset whose complement is negligible.* In other words, one tries to prove that the solution, or the minimizer u, is regular in some specified sense in an open subset $\Omega_u \subset \Omega$ such that $|\Omega \setminus \Omega_u| = 0$; the set

$$\Sigma_u := \Omega \setminus \Omega_u \tag{4.1}$$

is called *the singular set of* u. For this reason partial regularity is sometimes called *almost everywhere regularity.*

The first instance of such approach I am presenting is given by the following partial regularity analog of Theorem 2.5:

Theorem 4.3. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$ under the assumptions* (2.16) *with $\mu > 0$. Then there exists an open subset $\Omega_u \subset \Omega$ such that $|\Omega \setminus \Omega_u| = 0$ and $Du \in C^{0,\alpha/2}_{\text{loc}}(\Omega_u, \mathbb{R}^{nN})$.*

On the other hand, the analog of Theorem 2.7 is the following

Theorem 4.4. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a weak solution to the system* (2.8) *under the assumptions* (2.20) *with $\mu > 0$. Then there exists an open subset $\Omega_u \subset \Omega$ such that $|\Omega \setminus \Omega_u| = 0$ and $Du \in C^{0,\alpha}_{\text{loc}}(\Omega_u, \mathbb{R}^{nN})$.*

These two theorems have no analog in the degenerate case $\mu = 0$, unless some further structure assumptions are added, as we shall see in subsection 4.9 below. The singular set $\Sigma_u = \Omega \setminus \Omega_u$ in such theorems is identified by the equality

$$\Sigma_u = \Sigma_u^0 \cup \Sigma_u^1, \tag{4.2}$$

where

$$\Sigma_u^0 := \left\{ x \in \Omega \colon \liminf_{\varrho \searrow 0} \fint_{B(x,\varrho)} |Du(y) - (Du)_{x,\varrho}|^p \, \mathrm{d}y > 0 \right.$$
$$\left. \text{or } \limsup_{\varrho \searrow 0} |(Du)_{x,\varrho}| = \infty \right\},$$
$$\Sigma_u^1 := \left\{ x \in \Omega \colon \liminf_{\varrho \searrow 0} \fint_{B(x,\varrho)} |u(y) - (u)_{x,\varrho}|^p \, \mathrm{d}y > 0 \right.$$
$$\left. \text{or } \limsup_{\varrho \searrow 0} |(u)_{x,\varrho}| = \infty \right\}.$$

The reason for (4.2) is very basic: the partial regularity technique leading to Theorems 4.3 and 4.4 is actually a *linearization technique.* Let me sketch it in a simple case, the one of systems of the type $\operatorname{div} a(Du) = 0$, when we actually have $\Sigma_u = \Sigma_u^0$. Looking at the proofs, one realizes that the regular points x, that is the points the gradient is Hölder continuous in a neighborhood of, are basically those satisfying for a radius $r > 0$,

$$\fint_{B(x,r)} |Du(y) - (Du)_{x,r}|^p \, dy \leqslant \varepsilon, \tag{4.3}$$

and $\varepsilon > 0$ is suitably small. A regularity condition such as (4.3) is usually referred to as an "ε-regularity criterion". The original solution u is compared, in the ball $B_r \equiv B(x,r)$, with the solution $v \in W^{1,p}(B_r, \mathbb{R}^N)$ of the auxiliary, linearized system $\mathrm{div}(D_z a((Du)_{x,r})Dv) = 0$, which is a linear elliptic system with constant coefficients. Therefore the comparison map v is smooth, and enjoys good a-priori estimates. Here $(Du)_{x,r}$ is the average of Du over B_r. Then the next step is to make sure that u and v are, in some integral sense, close enough to each other in order to make u inherit the regularity estimates of v. This is achieved if the original system is "close enough" to the linearized one $\mathrm{div}(D_z a((Du)_{x,r})Dv) = 0$. Condition (4.3) serves to ensure this. From this argument the characterization in (4.3) naturally pops up, as well as the identity in (4.2). Such a linearization idea finds its origins in Geometric Measure Theory, and more precisely in the pioneering work of De Giorgi [78] on minimal surfaces, and of Almgren [12] for minimizing varifolds, and was first implemented by Morrey [256] and Giusti & Miranda [167] for the case of quasilinear systems $\mathrm{div}(a(u)Du) = 0$. Great impulse to the study of partial regularity of solutions to systems and minima of functionals, was initially given by the study of harmonic mappings and related elliptic systems, carried out in the papers by Hildebrandt & Kaul & Widman [182], [183]. For the completely non-linear case we have today different methods to implement the local linearization scheme described above: the hard, "direct method" applied by Giaquinta & Modica [160] and Ivert [187], [188]; the indirect one via blow-up techniques, implemented originally in the cited papers of Morrey and Giusti & Miranda, and then recovered directly for the quasiconvex case by Evans, Acerbi, Fusco, Hutchinson, and Hamburger [115], [142], [2], [173], [175]; see also Theorem 4.11, and the comments below. Finally, the technique I like most, the "A-approximation method", once again first introduced in the setting of Geometric Measure Theory by Duzaar & Steffen [104] and applied to partial regularity for elliptic systems and functionals by Duzaar & Gastel & Grotowski [94], [92], see also the nice survey [97]. This method re-exploits the original ideas that De Giorgi introduced in his treatment of minimal surfaces [78], providing a neat and elementary proof of partial regularity. It also shows that the heavy tool of reverse Hölder inequalities originally used in the papers of Giaquinta & Modica and Ivert can be actually completely avoided. The linearization is indeed implemented via a suitable variant, for systems with constant coefficients, of the classical "Harmonic approximation lemma" of De Giorgi, see also [283] and Subsection 4.9 below. The foregoing rough explanation also suggests why we have no analog of Theorem 4.4 in the case $\mu = 0$: when linearizing the system about the gradient average $(Du)_{x,r}$, it may happen that $(Du)_{x,r}$ is near the origin, or even zero, so that the linearized systems itself loses its ellipticity and regularizing properties, and in the end no comparison argument takes place. In such degenerate cases more accurate comparison procedures must be followed, and under additional structure assumptions on the integrand F, see Subsection 4.9 below.

4.3. Lack of low order partial regularity

Although someone may think differently, partial regularity theory for both systems and functionals is still widely incomplete; a whole partial regularity analog of the low regularity theory, for instance in the spirit of Subsection 2.2, is yet missing. Let me mention the following problem, which is amazingly still open: consider an elliptic system of the simple type

$$\operatorname{div} a(x, Du) = 0$$

satisfying (2.20), where this time we are not requiring (2.17) but just asking $\omega(\cdot)$ to vanish at zero. In other words, the dependence of $a(x, z)$ upon x is just continuous, rather than Hölder continuous. It is natural to ask whether there exists an open subset $\Omega_0 \subset \Omega$ such that $|\Omega \setminus \Omega_u| = 0$, and such that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega_0, \mathbb{R}^N)$ for some $\alpha > 0$ or, eventually, for any $\alpha < 1$. Note that in the case of equations and scalar functionals $N = 1$, this is known, with no singular set: $\Omega = \Omega_u$; compare Subsection 2.2. *The answer to such a basic issue is at the moment not known* except in certain low dimensional cases, as correctly shown by Campanato, see [52] and the survey [55]. It is interesting to know that Campanato himself gave a "proof of the result" in the general case, which revealed to be completely wrong [54]. More generally, no partial regularity result of any type is known for solutions to such a system. A similar problem exists of course for functionals, just considering (2.16) without requiring (2.17).

4.4. The size of the singular set

After getting Theorems 4.3 and 4.4, the next issue is of course trying to prove that Σ_u is not only negligible, but in some sense "smaller", if not empty at all. A way to do this is to give an upper estimate for the Hausdorff dimension $\dim_{\mathcal{H}}(\Sigma_u)$ of Σ_u. For basic systems of the type

$$\operatorname{div} a(Du) = 0, \tag{4.4}$$

and therefore for simple functionals of the type (3.6), it is possible to prove that

$$\dim_{\mathcal{H}}(\Sigma_u) < n - 2. \tag{4.5}$$

See for instance [52]. In particular, when $n = 2$, the singular set is empty. In the case of systems (2.8), assuming that both a_v and a_x exist and that the solution u is a-priori continuous, Giaquinta & Modica [160] proved (4.5) again. The problem in the general case (2.10) remained open (stated in [152], page 191, and [160], page 115) since [187], [160]. The first results in this direction can be found in my papers [248], [249], which I will summarize now, also taking into account some later improvements in [206], [98]. I shall start with the following result, essentially contained in [249]:

Theorem 4.5. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a weak solution to the system* (2.8) *under the assumptions* (2.20), *with $\mu > 0$. Then*

$$\dim_{\mathcal{H}}(\Sigma_u) \leqslant n - \min\{2\alpha, s - p\}, \tag{4.6}$$

where $s > p$ is the higher integrability exponent appearing in Theorem (4.2). *Moreover, if $n \leqslant p + 2$, or if the solution u is already locally Hölder continuous in Ω, then*

$$\dim_{\mathcal{H}}(\Sigma_u) \leqslant n - 2\alpha. \tag{4.7}$$

Finally, when $p = 2$ the previous inequalities become strict and, in the latter case, $\mathcal{H}^{n-2\alpha}(\Sigma_u) = 0$.

As stated in Theorem 4.2, the number s can be explicitly quantified, essentially in terms of the ellipticity ratio L/ν; see [44], [197], [288] and the discussion immediately after Theorem 4.2. Getting information on the Hausdorff dimension of certain branch, or removable sets, by estimating higher integrability exponents, is a strategy typically followed in the theory of quasiconformal mappings, where the role of the ration L/ν is played by the quasiconformality constant K of a quasiconformal mapping $f \in W^{1,n}(\Omega, \mathbb{R}^n)$:

$$|Df(x)|^n \leqslant K \det(Df(x)),$$

see [44], [190], [47]. In the last part of Theorem 4.5 the restriction to the case $p = 2$ appears to be technical, and I hope in the future someone will achieve the strict inequality for any $p > 1$. Results are also available for complete systems of the type in (2.2), see again [249]. There is also a further result from [248]:

Theorem 4.6. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a weak solution to the simpler system*

$$\operatorname{div} a(x, Du) = 0 \tag{4.8}$$

under the assumptions (2.20) *with $\mu > 0$, suitably recast for this case. Then*

$$\dim_{\mathcal{H}}(\Sigma_u) \leqslant n - 2\alpha. \tag{4.9}$$

Finally, when $p = 2$ this inequality becomes strict, and $\mathcal{H}^{n-2\alpha}(\Sigma_u) = 0$.

Comments are in order. Let me start from the last theorem. Estimate (4.9) tells us that the possibility of "reducing" the dimension of the singular sets is determined in a quantitative way by the regularity with respect to the coefficients: the more regular $a(x, z)$ is with respect to x, the better the estimate becomes. It is a sort of Schauder estimate for the singular set, and it agrees with (4.5), obtained assuming, among other things, differentiability with respect to x, that is, roughly, $\alpha = 1$. Such a viewpoint helps to give an intuitive explanation to estimate (4.6), valid in the general case. When dealing with a complete vector field of the type $a(x, u, z)$, the system (2.8) can be viewed as $\operatorname{div} b(x, Du) = 0$, where $b(x, z) \equiv a(x, u(x), z)$. At this point the Hölder continuity of $x \to b(x, z)$ is lost, since $u(x)$ may exhibit high irregularity, compare Subsection 3.2. Nevertheless, the fact that $u(x)$ is actually a solution to the system comes into play again via Theorem 4.2, and the L^s-integrability of Du serves to bound in a suitable way the oscillations of $u(x)$. Accordingly, in the low dimensional case $n \leqslant p+2$, it is possible to prove that u is Hölder continuous outside a closed subset of Hausdorff dimension less than $n-2$, and eventually we can recover the full estimate (4.7) again. The same obviously applies when $u(x)$ is a-priori assumed to be everywhere Hölder continuous. The technique for proving estimates (4.6)–(4.9) rests on the simple observation that the Hölder continuity dependence of the vector field $a(x, u, z)$ can be read as a fractional differentiability. Therefore, applying a variant of the standard difference quotient method technique [267], via suitable test functions and in combination with Gehring's lemma, it is possible to prove that the gradient is in a suitable fractional Sobolev space. In turn, this implies the estimate for the singular sets via abstract measure theoretical arguments. For details see [248], [249]. Extensions to the case of systems with Dini continuous coefficients, both with respect to partial regularity and to the singular sets dimension estimates, are also possible [91], [305], [93]. In this case more general Hausdorff measures come into play, more precisely those generated using Carathéodory's construction via a gage (generating) function which is not of power type; see [93] and references for more. The results of Theorems 4.5 and 4.6 have been extended by Kronz [212] to the case of higher order elliptic systems.

The problem of estimating the Hausdorff dimension in the case of minima also remained open since it was put forward in the papers [157], [158], [188], and even in the favorable case of C^∞ dependence of the integrand $F(x, v, z)$ with respect to (x, v); see the comments below, after Theorem 4.8. It was not clear whether $|\Sigma_u| = 0$ was already optimal or not, and the issue was raised several times: see for instance [153], open problem in Section 3, [154] comments in Section 4, and [156], open problem (a), page 117. This problem has been settled by Kristensen & myself in [206], where we have proved that *partial regularity in the sense of* $|\Sigma_u| = 0$ *is never optimal.* The first result is the analog of Theorem 4.6:

Theorem 4.7. *Let* $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ *be a local minimizer of the functional* $\mathcal{F}$ *under the assumptions* (2.16) *with* $\mu > 0$*. Then*

$$\dim_{\mathcal{H}}(\Sigma_u) \leqslant n - \min\{\alpha, s - p\}, \tag{4.10}$$

where $s > p$ *is the higher integrability exponent appearing in Theorem* 4.1*. Moreover, if* $n \leqslant p + 2$*, we have*

$$\dim_{\mathcal{H}}(\Sigma_u) \leqslant n - \alpha.$$

Finally, when $p = 2$ *the previous inequalities become strict and, in the latter case,* $\mathcal{H}^{n-\alpha}(\Sigma_u) = 0$*.*

Again, the number s essentially depends on L/ν, and therefore the bound on the Hausdorff dimension given in (4.10) can be made explicit, and quantified in terms of L/ν. Considerations analogous to those already made for the case of systems apply here. Just observe that, exactly as in Theorem 4.3 as compared to Theorem 4.4, there is a loss in the estimate, $n - \alpha$ instead of $n - 2\alpha$, when passing from solutions to systems to minimizers of functionals. Nevertheless, the bound in (4.6) can be recovered for minimizers in certain special cases [208]. It is interesting to point out that for functionals with a special structure such as

$$\int_\Omega a(x, u)|Du|^p \, \mathrm{d}x, \tag{4.11}$$

the bound can be improved up to $n - p$, see [157], [143]. It is also important to note that, both for systems and functionals, the singular set is empty in the two-dimensional case, at least when $p \geqslant 2$; see Section 9 in [206] and comments in Remark 2.2 from [98].

As far as the dependence on u is concerned, a sort of analog of the result of Theorem 4.6 is available:

Theorem 4.8. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional*

$$\int_\Omega f(x, Dv) + g(x, v)\, \mathrm{d}x \tag{4.12}$$

under the assumptions (2.16) *on f with $\mu > 0$, and provided $g\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a bounded, measurable function which is α-Hölder continuous with respect to the second variable, uniformly with respect to the first. Then*

$$\dim_{\mathcal{H}}(\Sigma_u) \leqslant n - \alpha. \tag{4.13}$$

Finally, when $p = 2$ this inequality becomes strict and $\mathcal{H}^{n-\alpha}(\Sigma_u) = 0$.

An explanation for the last improvement as compared to the estimate in (4.10) is that now the "disturbing" presence of $u(x)$ is decoupled by the regularizing term, that is the one containing Du. It is interesting to see that g can be taken to be only measurable in x; in that case partial regularity of minima has been proven in [173]. The theorem is a particular case of the ones contained in [206], where more general splitting structures of the type in (4.12) can be considered. The idea for proving Theorems 4.7 and 4.8 is again to show that Du is in a fractional Sobolev space, but this time the implementation must be completely different. In fact, the functionals under consideration do not possess the related Euler-Lagrange system, and it is not possible to use any test function technique. On the contrary, in [206] we introduce a new "variational difference quotient method", based on the minimality of u, and a delicate iteration/interpolation procedure in the setting of fractional Sobolev spaces. The basic idea is the following: since one cannot use the Euler-Lagrange system of the functional, one considers the Euler-Lagrange systems of certain differentiable functionals, obtained from the original one by a freezing procedure; in turn, these can be differentiated, and the related estimates are transferred to the original minimizer by a comparison argument. The final effect is an "indirect differentiation" of the original functional. Note that, as mentioned above, even assuming C^∞-regularity of $F(x, v, z)$ the situation does not improve: provided it does exist, the Euler-Lagrange system of the functional $\mathcal{F}$ is in general non-homogeneous, of the type (2.3), with a right-hand side with critical growth, i.e.: $|F_u(x, u, Du)| \leqslant L(1 + |Du|^p)$. It is not possible to use the usual difference quotients technique via test functions unless the solution is bounded, with a suitably small L^∞-norm. On the other hand, this is not the case in general, as we have seen in Subsection 3.4.

In my opinion there remains an interesting open problem to discuss the optimality of the Hausdorff dimension estimates contained in Theorems 4.5–4.8, eventually finding minima and solutions with large singular sets.

There are of course plenty of further problems arising in the analysis of singular sets: Are they rectifiable? Do they have additional geometric structures? As for the rectifiability of singular sets, in the case of harmonic maps and of minimal surfaces, the reader should have a look at the beautiful works of Simon [283], [282].

4.5. ω-minima and their singular sets

The techniques introduced to treat the singular sets of non-differentiable functionals also apply to the so called ω-minima. A map $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ is an ω-minimizer of the functional $\mathcal{F}$ in (2.1) under the assumption (2.11), if and only if

$$\int_{B_R} F(x, u(x), Du(x))\, \mathrm{d}x \leqslant [1 + \omega(R)] \int_{B_R} F(x, v(x), Dv(x))\, \mathrm{d}x \tag{4.14}$$

for any $v \in W^{1,p}(B_R, \mathbb{R}^N)$ such that $u - v \in W_0^{1,p}(B_R, \mathbb{R}^N)$, where $\omega\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing, concave function satisfying $\omega(0) = 0$, and $B_R \subset\subset \Omega$ is an arbitrary ball of radius R. ω-minima are sometimes called *almost minimizers.* Clearly, a minimizer is also an ω-minimizer, and the two classes are strictly different. The interest in ω-minima is motivated by the fact, originally observed in the setting of Geometric Measure Theory [13], [46], [14], that in many situations *minimizers of constrained problems* can be realized as *ω-minimizers of unconstrained problems.* Notably, the solutions to obstacle problems and to volume constrained problems are ω-minima, where the function $\omega(\cdot)$ is determined by the properties of the constraint [18], [92]. The regularity theory for minima extends, sometimes under suitable decay assumptions on the function $\omega(\cdot)$ to ω-minima in a quite satisfactory manner. For instance, Theorem 2.3 extends to any ω-minimizer, as shown in [89], [112], by the proof is far from being trivial; see also Chapter 7 from [165] for a proof of the result. In the vectorial case $N > 1$, assuming that the function $\omega(\cdot)$ satisfies (2.17), partial regularity of minima in the sense of Theorem 4.3 follows; see [92], [99], [207] for a proof. Therefore the problem of estimating the Hausdorff dimension of the singular set of ω-minima naturally arises. For the sake of simplicity I will restrict myself in the following to discussing simpler functionals of the type (3.6), whereas several results are available in the case of the complete ones as in (2.1) too. Once again the problem is the Euler-Lagrange system; even when it exists—and in the case of $\mathcal{F}_s$ it actually does!—it cannot be used just because ω-minima do not satisfy it, unless they are real minima. Nevertheless, using the comparison method described after Theorem 4.8 we have the following result:

Theorem 4.9. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be an ω-minimizer of the functional $\mathcal{F}_s$ under the assumptions* (2.16) *with $\mu > 0$, and assume that $w(\cdot)$ satisfies* (2.17)*. If*

Σ_u *denotes the singular set of* u *in the sense of* (4.1), *then*

$$\dim_{\mathcal{H}}(\Sigma_u) \leqslant n - \alpha, \quad \alpha \in (0,1]. \tag{4.15}$$

For this result, which improves in certain cases the one in [204], I refer to the forthcoming paper [208], to which I also refer for results in the case of functionals (2.1). Note that estimate (4.15) states that the better ω decays with respect to (2.17), the smaller is the singular set, in perfect accordance with the phenomenon already recorded in Theorems 4.5 and 4.7. Remarkably, and actually not by chance, estimates (4.15) and (4.13) are the same.

4.6. Boundary problems

I will now briefly report on some recent developments concerning Dirichlet problems, and on partial regularity on the boundary of solutions to non-linear elliptic systems; these are related to the above singular sets estimates. Let me consider the following Dirichlet problem associated with the system (2.8) under the assumptions (2.20):

$$\begin{cases} \operatorname{div} a(x,u,Du) = 0 & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega, \\ u_0 \in C^{1,\alpha}(\bar{\Omega}, \mathbb{R}^N), \quad \partial\Omega \text{ is } C^{1,\alpha}\text{-regular.} \end{cases} \tag{4.16}$$

One can ask if partial regularity carries up to the boundary. In fact, it is possible to prove a boundary regularity criterion ensuring that a boundary point $x_0 \in \partial\Omega$ is regular in the sense that Du is Hölder continuous in a neighborhood of x_0, in the relative topology of $\bar{\Omega}$. Exactly as for (4.3), this is the case if and only if for some small positive number ε we have

$$\fint_{B(x_0,R)\cap\Omega} |Du - (Du)_{B(x_0,R)\cap\Omega}|^p \, \mathrm{d}x < \varepsilon. \tag{4.17}$$

For such a result see [174], [169], [28], [29]. Unfortunately, condition (4.17) does not yield the existence of regular boundary points, since it is verified a.e. with respect to the Lebesgue measure, while the boundary $\partial\Omega$ is a null set. The problem of finding the existence *of even regular boundary point* has remained open, see comments at page 246 of [152], while, on the other hand, the existence of irregular boundary points has been known for a while [151], and even for systems with a special, simpler structure; what a bizarre situation! Moreover, this gap is in sharp contrast to what happens in the case of elliptic equations, where full regularity carries up to the boundary [159], and in the case of quasi-linear elliptic systems, i.e. those of the form

$\operatorname{div} a(x,u)Du = 0$, where a.e. boundary point (in the sense of the usual surface measure) is regular [65], [170], [19], [29]. A first answer to the problem was given in [98], building on the work in [248], [249]. The idea is to carry estimate (4.7) up to the boundary; then assuming that α is suitably large we have that a.e. boundary point is regular. We have indeed

Theorem 4.10. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ be a weak solution to* (4.16) *under the assumptions* (2.20), *and assume that*

$$\alpha > \frac{1}{2}. \tag{4.18}$$

Moreover, assume that either $n \leqslant p+2$, or $a(x,u,Du) \equiv a(x,Du)$. Then almost every boundary point $x \in \partial\Omega$, in the sense of the usual surface measure, is a regular point, i.e. the gradient is $C^{0,\alpha}$-regular in a neighborhood of x relative to $\bar{\Omega}$. Moreover, when $p = 2$, we can allow the borderline case $\alpha = 1/2$.

If fact, under the assumption (4.18), we have that $n - 2\alpha < n - 1$ and then the Hausdorff dimension of the boundary singular set is strictly less that the dimension of the boundary itself; in particular, the existence of regular boundary points follows. The technique used in [98] is different from that in [248], [249], and rests on a new, indirect way of treating fractional difference quotients, dealing with them by a new comparison argument based on convolutions. Such a technique is perhaps interesting in itself and could find applications elsewhere. *There remains the problem to discuss the existence of regular boundary points when $\alpha \in (0, 1/2)$*, a case which is excluded by the methods adopted for Theorem 4.10. A version of Theorem 4.10 valid for minima of a large class of integral functionals, with an integrand which is convex in the gradient variable, is given in the forthcoming paper [208]. In [208] we extend to a family of very general functionals the existence results of Jost & Meier [195], which were valid under a very peculiar structure assumption on the integrand F, which was of the type (4.11) with $p = 2$; for the case $p \neq 2$ and again functionals as in (4.11) see [95]. The boundary ε-regularity criterion for minima in the sense of (4.17) was given by Kronz [211], and directly for ε-minima of quasiconvex integrals. See also the discussion at the end of Subsection 4.8.

4.7. Conditions for everywhere regularity

A fundamentally important and open problem is clearly the one of identifying classes of functionals for which everywhere $C^{1,\alpha}$, or even just continuity, of minimizers occurs. The same problem arises for solutions to systems. In other words: are there additional structure assumptions on the integrand F, or on the vector field a, under which the singular set is void? Up to now, the only known structure preventing the formation of singularities for minimizers is the one first identified in the fundamental work of K. Uhlenbeck [301]. It prescribes that

$$F(x, v, z) \equiv F(z) = g(|z|) \tag{4.19}$$

for a suitable function $g := [0, \infty] \to [0, \infty]$ such that (2.16) is still satisfied. So, the dependence of the gradient must occur directly via the modulus $|Du|$, which makes, in a sense, the functional "less anisotropic" and rules out singularities of minima, see [266], [291], [290]. The one in (4.19) is sometimes called "Uhlenbeck structure" [31]. In the case of systems, the counterpart of (4.19) is

$$a_i^\alpha(x, v, z) \equiv a_i^\alpha(z) = h(|z|)z_i^\alpha, \quad \alpha \in \{1, \dots, N\}, \ i \in \{1, \dots, n\}, \tag{4.20}$$

and $h := [0, \infty] \to [0, \infty]$ is once again a suitable function such that (2.20) is satisfied. An extension to Uhlenbeck's results can be found in [143], [171], to which I refer for proofs and references.

The challenge is nowadays to identify new structures, different from the ones in (4.19)–(4.20), forcing everywhere regularity. In the case of quasi-linear systems some conditions can be found in [201]. Assuming that F_{zz} does not have large oscillations, it is also possible to prove everywhere regularity: this is the so called "linearity condition", see [76] and related references.

4.8. Quasiconvexity

Up to now, I have dealt with convex functionals. Convexity is suitable to ensure lower semicontinuity for variational integrals, and therefore existence of minima. In the vectorial case there is anyway another condition, much weaker than convexity, which is sufficient for lower semicontinuity, and actually necessary under certain natural assumptions: this is the so called quasiconvexity. It makes therefore sense to ask for regularity of minima under such a condition. For simplicity's sake, from now on I shall confine myself to considering simpler variational integrals as in (3.6). A function $F : \mathbb{R}^{nN} \to \mathbb{R}$ is quasiconvex iff

$$\int_{(0,1)^n} [F(z_0 + D\varphi) - F(z_0)] \, \mathrm{d}x \geqslant 0 \tag{4.21}$$

for every $z_0 \in \mathbb{R}^{nN}$ and every $\varphi \in C^\infty((0, 1)^n, \mathbb{R}^N)$, with compact support in $(0, 1)^n$. Such a definition deserves comments. First, by a covering argument $(0, 1)^n$ can be replaced by any other open subset $\Omega \subset \mathbb{R}^n$; moreover, convex functions are trivially quasiconvex via Jensen's inequality, nevertheless the two definitions are strictly different. Quasiconvexity states that affine functions of the type $w_0 + \langle z_0, x\rangle$,

$w_0, z_0 \in \mathbb{R}^{nN}$, are minimizers of the functional $\mathcal{F}_s$ in (3.6), in their Dirichlet class. A large class of quasiconvex functions, strictly intermediate between the one of convex ones, and the one of quasiconvex themselves, is the class of the so called polyconvex functions [23], [27], [73], [124]. In the special case $u\colon \Omega \to \mathbb{R}^n$, $n = N$, these are integrands g of the form

$$\int_\Omega g(Dv, \operatorname{Ad} Dv, \det Dv)\, \mathrm{d}x \tag{4.22}$$

where g is a convex function of all its arguments, and $\operatorname{Ad} Dv$ stands for the matrix of all the minors of Dv.

The difficulty in treating quasiconvex functions largely stems from the non-local nature of quasiconvexity, as is immediately clear from definition (4.21). This point is not fixable: indeed, proving a fundamental and longstanding conjecture of Morrey, Kristensen [202] showed that there is no local condition characterizing quasiconvexity. The notion of quasiconvexity was introduced by Morrey [254], who first identified its connection to lower semicontinuity; the first general lower semicontinuity result is contained in the seminal paper of Acerbi & Fusco [1], where it is shown, for instance, that an autonomous quasiconvex functional as in (3.6), satisfying (2.11), is weakly lower semicontinuous in $W^{1,p}$ if and only if $F(z)$ is quasiconvex. In this paper the authors also introduced a number of important techniques for treating general lower semicontinuity problems in the Calculus of Variations. See also [141], [242], [71] for related existence results, while for further important progress on lower semicontinuity issues, see [26], [233], [125], [203]. Quasiconvexity plays an important role in the context of non-linear elasticity, as discussed in the fundamental work of Ball [23]. For further basic information on quasiconvexity, the reader is referred to [72], [165], [261].

The partial regularity theory for quasiconvex functionals was initiated first by Evans [115], who used the following reinforcement of the definition in (4.21):

$$\nu \int_{(0,1)^n} (1+|z_0|^2+|D\varphi|^2)^{(p-2)/2}|D\varphi|^2\, \mathrm{d}x \leqslant \int_{(0,1)^n} [F(z_0+D\varphi)-F(z_0)]\, \mathrm{d}x, \tag{4.23}$$

that he called *uniform strict quasiconvexity*, and that serves to provide a sort of non-degenerate quasi-convexity. For instance, if F is a convex, C^2-function in the scalar case $N = 1$, then (4.23) implies the left-hand side inequality in $(2.16)_3$ for a different $\nu > 0$, as can be retrieved in [122]. We have the following partial regularity result:

Theorem 4.11. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}_s$ in* (3.6) *such that $F(z)$ is a C^2-function satisfying* (2.11) *and* (4.23). *Then there exists an open subset $\Omega_u \subset \Omega$ such that $|\Omega \setminus \Omega_u| = 0$, and $Du \in C^{0,\alpha}_{\mathrm{loc}}(\Omega_u, \mathbb{R}^{nN})$ for any $\alpha \in (0,1)$.*

This theorem was first obtained by Evans [115] under more restrictive assumptions. The version here is due to Acerbi & Fusco [2] in the case $p \geqslant 2$, and to Carozza & Fusco & myself in the case $1 < p < 2$ [56]. Note that for quasiconvex functions the case $1 < p < 2$ cannot be treated via duality methods starting from the case $p \geqslant 2$ as in the convex case [171], and it requires a new, direct technical approach. Again observe that here, unlike in $(2.16)_3$, no upper bound on F_{zz} is assumed; this is the main important contribution of [2], which of course extends also to the convex case, see for instance [204] for comments. Extensions to complete functionals of the type in (2.1) are possible, and actually sharply done, while alternative proofs via the A-approximation method can be found in [92], [96]. A weakening of growth conditions in a more general case is proposed by Hong [184], while a localization regularity theorem for minima of functionals which are not necessarily everywhere quasiconvex was proved by Acerbi & Fusco [3]. More recently, the partial regularity of *strong local minimizers* of quasiconvex functionals has been proved by Kristensen & Taheri [209]. As far as the lower order regularity is concerned, there are very few results available, all of them prescribing additional structure assumptions on the integrand F. For instance assuming that the integrand F is only asymptotically near (at infinity) to a strongly elliptic quadratic form, Chipot & Evans [61] proved the Lipschitz continuity of minima; for this and related results see after Theorem 4.12. Further remarkable extensions, higher integrability results for minimizers, and even minimizing Young measures, are in the work of Dolzmann & Kristensen [90].

Note that a partial $C^{1,\alpha}$-regularity theory is also available for certain polyconvex functionals as in (4.22); for instance, when $n = N \geqslant 2$, the following functional can be treated:

$$\int_\Omega (|Dv|^p + |\mathrm{Ad}\, Dv|^p + |\det Dv|^p)\, \mathrm{d}x, \quad p > n-1.$$

This time the minimization problem must be settled in appropriate function spaces; the reader is referred to [144], [145], [114]. *An important open problem remains, namely the one of proving partial regularity of minima of the functional in* (4.22) *under the "realistic blow-up condition"*

$$g \equiv g(z, \det z) \nearrow \infty \quad \text{when} \quad \det z \to 0, \tag{4.24}$$

which is of importance in non-linear elasticity; see the very nice review paper by Ball [24]. The minimizer is this time in the class of orientation preserving maps v, i.e. $\det Dv > 0$. An interesting maximum principle in this case has been recently proved by Leonetti [220]; if the minimization of the functional in (4.22) is performed in a Dirichlet case with bounded boundary data, then the minimizer is itself bounded, provided the rate of blow-up in (4.24) is suitably controlled and not too fast. On such a problem, see also some interesting attempts in [139], [140], [116], [128].

It is important to understand that, due to the non-local character of quasiconvexity, the regularity theory for minima of quasiconvex functionals is much more delicate than that for convex ones in many respects. Indeed, roughly speaking, while convexity allows to compare minimizers with a lot of other maps, quasiconvexity restricts the possibility of comparison arguments to locally affine-looking maps only. Moreover, the use of the Euler-Lagrange systems is strictly forbidden! This is clearly seen when looking at critical points of the functional $\mathcal{F}_s$ in (3.6). Indeed, when considering a convex functional of the simple type (3.6), Theorem 4.3 holds for minimizers but, via Theorem 4.4, it immediately extends to critical points, that is solutions to the Euler-Lagrange system (3.7). This is not the case for quasiconvex functionals. Indeed, even assuming that F is a smooth, uniform and strict quasiconvex function as in Theorem 4.11, and with $p = 2$, Müller & Šverák [262] provided amazing counterexamples of non-minimizing solutions u to the Euler-Lagrange system (3.7) that are not differentiable on any open subset of Ω. The problem of determining lower order irregularity of critical points remains anyway still open, since the solutions exhibited by Müller & Šverák are still Lipschitz continuous. The example of Müller & Šverák is completely different from the ones considered in Section 3, and is based on a delicate construction resting on the use of the so-called Tartar's "T_4-configuration" and of Gamow's convex integration theory [168]; for more on the issue see also their paper with Kirchheim [200]. The work of Müller & Šverák is nowadays generating massive developments: let me mention the remarkable extension of their results to the case of polyconvex functionals by Székelyhidi [292], using this time "T_5-configurations"; see also [293] for a striking two-dimensional result. Moreover, Bevan [32], [33] has obtained two dimensional examples of non-C^1-minimizers of strictly polyconvex functionals, that is, when the function g appearing in (4.22) is strictly convex. Compare also the recent work by Phillips [268].

At the moment no estimate for the Hausdorff dimension in the general case is available for minima of quasiconvex functionals, and no analog of Theorem 4.7 has been proven yet, even for simpler functionals of the type (3.6), for which estimate (4.5) is on the other hand available in the convex case. This is essentially due to the fact that while in the convex case the dimension estimates are obtained by using the Euler-Lagrange system and differentiating it in some direct or indirect way, here, as noted above, such a tool does not yield regularity results in itself. Under additional assumptions, either on the structure of the integrand F or on the regularity of the minimizer u, a few first results, have been obtained by Kristensen & myself in [207]. We have indeed the following

Theorem 4.12. *Let $u \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}_s$ in* (3.6) *such that $F(z)$ is a C^2-function satisfying* (2.11) *and* (4.23) *with $p \geqslant 2$, a let $\Sigma_u := \Omega \setminus \Omega_u$ be the singular set of u, in the sense of Theorem* 4.11. *Then there exists a positive number*

$$\delta \equiv \delta(n, N, p, L/\nu, \|u\|_{W^{1,\infty}}) > 0, \tag{4.25}$$

independent of the minimizer u, such that

$$\dim_{\mathcal{H}}(\Sigma_u) \leqslant n - \delta. \tag{4.26}$$

The number δ appearing in (4.25) is in principle explicitly computable by carefully keeping track of the constants involved in the proof. It depends on the integrand F via two features only: first, the modulus of continuity of its second derivatives F_{zz}:

$$|F_{zz}(z_2) - F_{zz}(z_1)| \leqslant \gamma(|z_1| + |z_2|, |z_2 - z_1|) \quad \forall\, z_1, z_2 \in \mathbb{R}^{nN},$$

where $\gamma(\cdot,\cdot)$ is a non-decreasing, continuous and positive function such that $\gamma(\cdot, 0) = 0$; second, the associated "growth function"

$$G(M) := \sup_{|z| \leqslant M} \frac{|F_{zz}(z)|}{1 + |z|^{p-2}}, \qquad M \geqslant 0.$$

Interestingly, and surprisingly enough, the result of Theorem 4.12 extends to more general quasiconvex functionals of the type (2.1), once again assuming the Hölder continuity of the function $(x, y) \mapsto F(x, y, z)$ in the sense of $(2.16)_4$. In this case, on the contrary to what happened for Theorem 4.7, the number δ is still independent of the Hölder continuity exponent α in (2.17). On the other hand, we are assuming the minimizer u to be already globally Lipschitz continuous. Concerning such $W^{1,\infty}$-assumption, this is verified for a vast class of quasiconvex functionals of the type (3.6), which are "asymptotically near" the p-Laplacean functional. Indeed, assume that

$$\lim_{|z| \to +\infty} \frac{|D^2F(z) - D^2H(z)|}{|z|^{p-2}} = 0, \tag{4.27}$$

where

$$H(z) := (\mu^2 + |z|^2)^{p/2}, \quad \mu \in [0, 1].$$

A simple model example is given by

$$\int_\Omega (1+|Dv|^2)^{p/2} + g(Dv)\,\mathrm{d}x, \tag{4.28}$$

where $g\colon \mathbb{R}^{nN} \to \mathbb{R}^+$ is a C^2 and quasiconvex function (not necessarily strictly), such that $D^2g(z)/|z|^{p-2} \to 0$ when $z \to \infty$. For such classes of functionals we have local Lipschitz continuity of $W^{1,p}$-minimizers, and therefore it follows from Theorem 4.12 that for every $\Omega' \subset\subset \Omega$ there exists a positive number

$$\delta' \equiv \delta'(n, N, p, L/\nu, \mathrm{dist}(\Omega', \partial\Omega)) > 0$$

such that $\dim_{\mathcal{H}}(\Sigma_u \cap \Omega') \leqslant n - \delta'$. Finally, when assuming (4.27) and minimizing $\mathcal{F}_s$ in a prescribed Dirichlet class $u_0 + W^{1,p}_0(\Omega, \mathbb{R}^N)$ with $u_0 \in C^{1,\beta}(\Omega, \mathbb{R}^N)$, for some $\beta > 0$ and $\partial\Omega$ smooth enough, let us say C^2, then $W^{1,p}$-minimizers are globally Lipschitz continuous, and estimate (4.26) works with δ depending only on n, N, L/ν, p, $\partial\Omega$ and $\|u_0\|_{C^{1,\beta}}$. The interior Lipschitz continuity of minima under the additional assumption (4.27) has been obtained by Chipot & Evans [61] for $p = 2$, and extended to the case $p \geqslant 2$ in [162], [272], [137]. For the global regularity result see [135] when $p = 2$, and the recent, interesting paper of Foss [129] when $p \neq 2$.

Let me observe that Theorem 4.12 extends to ω-minima in the sense of Subsection 4.5 as well, and therefore improves the result of Theorem 4.9 when α is small; see again [207].

The methods of [207] also apply to solutions to the so-called quasimonotone systems, a notion independently introduced by Fuchs [134], Hamburger [172], and Zhang [306]. These are systems in divergence form of the type (4.4), where the ellipticity condition, that is the left-hand side inequality in $(2.20)_3$, is replaced by an integral, non-local condition similar to the one in (4.23), that is

$$\begin{aligned} \nu \int_{(0,1)^n} (1+|z_0|^2 + |D\varphi(x)|^2)^{(p-2)/2} |D\varphi(x)|^2 \,\mathrm{d}x \\ \leqslant \int_{(0,1)^n} \langle a(z_0 + D\varphi) - a(z_0), D\varphi\rangle \,\mathrm{d}x \end{aligned} \tag{4.29}$$

for every $z_0 \in \mathbb{R}^{nN}$ and every $\varphi \in C^\infty((0,1)^n, \mathbb{R}^N)$ having compact support. Partial regularity of weak solutions to quasimonotone systems holds in the sense of Theorem 4.11, which has been proved by Hamburger [172]. On the other hand, condition (4.29) is too weak to allow the application of any type of difference quotients method, and therefore for weak solutions to quasimonotone systems (4.5) cannot be derived, and no singular set estimate is known. In [207] Theorem 4.12 is seen to hold for weak solutions to quasimonotone systems too. For regularity and quasimonotonicity see also [134], [210], while for existence theorems see [306].

The technique employed for proving Theorem 4.12 is completely different from the freezing/comparison one used for Theorem 4.7, and in particular no use of fractional Sobolev spaces is made. On the contrary, we employ certain integral characterizations of potential spaces in combination with Caccioppoli's type inequalities in order to prove that the singular set Σ_u enjoys a property known in Geometric Measure Theory as "set porosity", see [246]. From this fact estimate (4.26) follows in a standard way, see also [275] for a wide discussion.

Let me close the subsection with an interesting open problem, regarding both the quasiconvex functionals and the quasimonotone systems. I recommend the reader to keep in mind the discussion in Subsection 4.6, where we have seen that the "ε-regularity" criterion on the boundary (4.17) also works for minima of quasiconvex functionals [211]. With some additional efforts this extends also to quasimonotone systems. As observed in Subsection 4.6 this does not imply the existence of regular boundary points. This time Theorem 4.12 does not help: carrying it up to the boundary yields no information. Indeed, in general δ is small, while we would need that $n - \delta < n - 1 = \dim_{\mathcal{H}}(\partial\Omega)$, when $\partial\Omega$ is smooth. Even worse, due to the set-porosity-techniques adopted, the method used in [207] provides a critical upper bound for δ: $\delta \leqslant 1$! Therefore, in strong contrast to the convex/elliptic case, *to establish the existence of regular boundary points for minima of quasiconvex integrals, and of solutions to quasimonotone systems, and eventually their almost everywhere regularity at the boundary, remains an open problem.*

4.9. Partial regularity, and degeneration

We have seen that in the vectorial case $N > 1$ no degenerate analog of Theorems 4.3 and 4.4 takes place, i.e. we cannot allow $\mu = 0$, while at the end of the same subsection a rough explanation of this is given in terms of the impossibility of linearizing when the gradient of the minimizer approaches 0. The problem of proving partial regularity for minima of degenerate (quasiconvex) functionals was raised in [154], Section 3. A first answer was given in [101] by Duzaar & myself, where we showed that assuming additional structure properties on the integrand yields partial regularity in degenerate cases too. Here I shall restrict myself, for the sake of simplicity, to the case of functionals of the type $\mathcal{F}_s$ in (3.6), and report on a special case of the results in [101] that work directly for quasiconvex integrals. Let me consider a quasiconvex, not necessarily strictly quasiconvex, C^2-function $g\colon \mathbb{R}^{nN} \to \mathbb{R}$ such that

$$0 \leqslant g(z) \leqslant L(1 + |z|^p) \tag{4.30}$$

and

$$\lim_{t \to 0^+} \frac{g_z(tz)}{t^{p-1}} = 0 \tag{4.31}$$

uniformly on the set $\{z \in \mathbb{R}^{nN} : \ |z| = 1\}$. Now, let me define the following *degenerate quasiconvex* functional:

$$\mathcal{DQ}(v) := \int_\Omega \nu |Dv|^p + g(Dv)\, \mathrm{d}x \tag{4.32}$$

with $\nu > 0$. Then we have

Theorem 4.13. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{DQ}$ under the assumptions* (4.31) *and* (4.32)*. Then there exists an open subset $\Omega_u \subset \Omega$ such that $|\Omega \setminus \Omega_u| = 0$, and $Du \in C^{1,\alpha}_{\mathrm{loc}}(\Omega_u, \mathbb{R}^n)$ for some $\alpha \in (0,1)$.*

For a more general result see again [101]. The proof of Theorem 4.13 offers an example of application of the "p-harmonic approximation lemma", obtained in [100], that extends the original De Giorgi's one [78], [283] to the case $p \neq 2$. I feel that in the context of the p-Laplacean theory, and in the one of harmonic mappings, it is of its own interest. I therefore find it worth reporting the statement here.

Theorem 4.14 (p-harmonic approximation lemma)**.** *Let B be the unit ball in $\mathbb{R}^n$. For each $\varepsilon > 0$ there exists a positive constant $\delta \in (0,1]$ depending only on n, N, p and ε such that the following is true: Whenever $u \in W^{1,p}(B, \mathbb{R}^N)$ with $\int_B |Du|^p\, \mathrm{d}x \leqslant 1$ is approximately p-harmonic in the sense that*

$$\left| \int_B |Du|^{p-2} Du \cdot D\varphi \, \mathrm{d}x \right| \leqslant \delta \sup_{B_\varrho} |D\varphi|$$

holds for all $\varphi \in C^1_c(B, \mathbb{R}^N)$, there exists a p-harmonic function $h \in W^{1,p}(B, \mathbb{R}^N)$ such that

$$\int_B |Dh|^p \, \mathrm{d}x \leqslant 1 \quad \textit{and} \quad \int_B |h - u|^p \, \mathrm{d}x \leqslant \varepsilon^p.$$

A function h is of course said to be p-harmonic when $\mathrm{div}(|Dh|^{p-2}Dh) = 0$. The idea for proving Theorem 4.13, inspired by the original strategy of Uhlenbeck [301], is that when the gradient Du of the minimizer is far from 0 in a quantitatively determined way, then one can linearize and prove partial regularity as discussed in Subsection 4.2. At this stage the linearization is achieved via the A-harmonic approximation method of Duzaar & Steffen [104], which is a variant of Theorem 4.14 for $p = 2$. On the contrary, when the gradient is near zero, so that the problem becomes really degenerate, one directly compares u with minimizers of the functional (2.18), which are regular by the results in [301], compare Subsection 4.7, and good regularity estimates for Du still follow. This time the comparison argument is achieved via Theorem 4.14, which is useful when treating truly degenerate situations. This last step is possible since (4.32) tells us that near $z = 0$, the integrand of $\mathcal{DQ}$ behaves essentially as $\nu|z|^p$. Finally, one shows that the classical harmonic (or A-harmonic) approximation lemma and the new p-harmonic one perfectly match in a suitable iteration procedure.

As a final observation let me mention that when $p = 2$, the parabolic analog of Theorem 4.14, and therefore of the original De Giorgi's lemma, has been obtained in [102]; *for the case $p \neq 2$ proving such a parabolic analogue remains an open problem.*

5. Irregularity strikes back

In this section, and in the next one, I shall restrict my attention to functionals of the type

$$\mathcal{F}(v) := \int_\Omega F(x, Dv)\,\mathrm{d}x. \tag{5.1}$$

We have seen that, while in the vectorial case $N > 1$ there is no hope to get everywhere regularity of minimizers for general variational integrals as in (2.1), at least in the scalar case $N = 1$, everywhere regularity in the interior of Ω is guaranteed under reasonable assumptions; this is essentially the content of Section 2. All the results in Section 2 follow assuming at least one common main condition, that is (2.11). Now let us take a look at the following functionals, where $1 < p < q$ are fixed numbers:

$$\mathcal{F}_1(v,\Omega) = \int_\Omega |Dv|^p \log(1 + |Dv|)\,\mathrm{d}x;$$

$$\mathcal{F}_2(v,\Omega) = \int_\Omega \sum_{i=1}^n a_i(x)|D_i v|^{p_i}\,\mathrm{d}x,$$

$$1 \leqslant a_i(x) \leqslant L, \quad 1 < p := p_1 \leqslant p_2 \leqslant \ldots \leqslant p_n =: q;$$

$$\mathcal{F}_3(v,\Omega) = \int_\Omega |Dv|^p + a(x)|Dv|^q\,\mathrm{d}x, \quad 0 \leqslant a(x) \leqslant L;$$

$$\mathcal{F}_4(v,\Omega) \equiv \mathcal{D}_{p(x)}(v) = \int_\Omega |Dv|^{p(x)}\,\mathrm{d}x, \quad 1 < p \leqslant p(x) \leqslant q;$$

$$\mathcal{F}_5(v,\Omega) = \int_\Omega |Dv|^{p(x)B(|Dv|)}\,\mathrm{d}x, \quad 1 < p \leqslant p(x) \leqslant q - \gamma, \quad 1 \leqslant B(|z|) \leqslant \gamma;$$

$$\mathcal{F}_6(v,\Omega) = \int_\Omega |Dv|^{p(x)} B(|Dv|)\,\mathrm{d}x,$$
$$1 < p \leqslant p(x) \leqslant q - q_1, \quad 1 \leqslant B(|z|) \leqslant L(1 + |z|^{q_1}).$$

None of the integrands corresponding to the functionals $\mathcal{F}_1$–$\mathcal{F}_6$ satisfies conditions (2.11) for any possible choice of the exponent $p \geqslant 1$. But all of them satisfy, for the correspondingly specified choice of the numbers (p,q) and suitable ν and L, the new, more general growth conditions

$$\nu|z|^p - L \leqslant F(x,z) \leqslant L(1 + |z|^q), \quad 1 < p < q. \tag{5.2}$$

Functionals satisfying conditions (5.2) and not meeting those in (2.11), are called functionals with (p,q)-*growth conditions.* Here I am following the terminology of Marcellini, who was the first to initiate a systematic study of such integrals in a series of seminal papers [235]–[239]. A related notion can be introduced for equations, but I am not going to deal with them in this paper. Before going on, let me point out some permanent assumptions. Due to the (p,q)-growth conditions satisfied by the integrand F, the following more general definition of minimality is usually adopted in this case: A map $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is a local minimizer of $\mathcal{F}$ iff $x \to F(x, Du(x)) \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_{\operatorname{supp}\varphi} F(x, Du)\,\mathrm{d}x \leqslant \int_{\operatorname{supp}\varphi} F(x, Du + D\varphi)\,\mathrm{d}x$$

for any $\varphi \in W^{1,1}(\Omega, \mathbb{R}^N)$ such that $\operatorname{supp}\varphi \subset \Omega$. From this definition it immediately follows that any local minimizer is in $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, because of the left-hand side inequality in (5.2). On the other hand, the same left-hand side inequality in (5.2) guarantees coercivity of $\mathcal{F}$ in $W^{1,p}(\Omega, \mathbb{R}^N)$, and therefore, provided the functional itself is lower semicontinuous in the weak topology of $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, direct methods of the Calculus of Variations guarantee the existence of minimizers in $W^{1,p}(\Omega, \mathbb{R}^N)$ with fixed boundary data. Lower semicontinuity under (p,q) growth conditions can be achieved by just assuming, for instance, that F is convex in the gradient variable, see [165], Chapter 4.

In order to show the impact of (p,q)-growth conditions on the regularity and/or irregularity of minima, I will start with examples of Marcellini, which elaborate upon previous counterexamples by Marcellini himself [234], and Giaquinta [155].

5.1. A first type of examples (Marcellini [236])

Marcellini considered a family of elliptic equations and integral functionals satisfying (p,q)-growth conditions, a particular case of which is

$$\mathcal{M}(v) := \int_\Omega \sum_{i=1}^{n-1} |D_i v|^2 + \frac{1}{2}|D_n v|^4 \,\mathrm{d}x,$$

and proved the existence of unbounded solutions provided

$$q > \frac{(n-1)p}{n-1-p}, \quad n > 2, \quad 1 < p < n-1. \tag{5.3}$$

For instance, when $n \geqslant 6$, the unbounded function

$$u(x) := \sqrt{\frac{n-4}{24}} \frac{x_n^2}{\sqrt{\sum_{i=1}^{n-1} x_i^2}}$$

is the unique minimizer of $\mathcal{M}$ in its Dirichlet class.

It is interesting to see that Marcellini's examples are concerned with degenerate integrals. For instance, when $|D_n u|$ approaches 0 the Euler-Lagrange equation of the functional $\mathcal{M}$

$$\sum_{i=1}^{n-1} D_{ii}u + \frac{1}{2}D_n(|D_n u|^2 D_n u) = 0$$

becomes degenerate elliptic, losing ellipticity in the x_n-direction. This point was fixed by Hong [185], who considered the regular, non-degenerate elliptic functional

$$\mathcal{H}(v) := \int_\Omega |Dv|^2 + \frac{1}{2}|D_n v|^4 \,\mathrm{d}x$$

having a regular integrand and exhibiting the following minimizer for $n \geqslant 6$:

$$u(x) := \sqrt{\frac{n-4}{24}} \frac{x_n^2}{\sqrt{\sum_{i=1}^{n-1} x_i^2}} - \frac{2}{n-2}\sqrt{\frac{n-4}{24}}\sqrt{\sum_{i=1}^{n-1} x_i^2}.$$

Hong's example is useful because it confirms the intuitive fact that for functionals with (p,q)-growth, problems mainly come from the behavior of the integrand $F(x,z)$ for large values of $|z|$. For a further discussion on counterexamples see the survey of Leonetti [219].

5.2. Second type of examples (Fonseca & Malý & myself [126])

These work for non-autonomous functionals of the type (5.1). A one-point, isolated singularity version of them had been previously obtained in [111], elaborating on some constructions of Zhikov [308] in the theory of Lavrentiev Phenomenon, see also Subsection 6.5 below. The examples show that, provided p and q are far enough from each other, depending on the dimension n, and under the condition of regularity of $x \mapsto F(x,z)$, the set of non-Lebesgue points of minimizers can be nearly as bad as any other $W^{1,p}$-function. Indeed, a well known measure theoretic result states that the set of non-Lebesgue points of (the precise representative of) a $W^{1,p}$-function has Hausdorff dimension not larger than the maximal dimension $n-p$, see for instance [165], Chapter 2. Here *it is possible to find a minimizer of a convex, regular, and scalar variational integral, whose set of non-Lebesgue points is a Cantor-type set of (nearly) maximal dimension.* Indeed, we have

Theorem 5.1. *For every choice of the parameters*

$$2 \leqslant n, \quad \alpha \in (0,\infty), \quad 1 < p < n < n+\alpha < q < \infty, \quad \varepsilon > 0, \tag{5.4}$$

there exist a functional

$$\mathcal{F}_{3.2}\colon\, u \mapsto \int_\Omega [(1+|Du|^2)^{p/2} + a(x)(1+|Du|^2)^{q/2}]\,\mathrm{d}x, \quad u \in W^{1,p}(\Omega), \tag{5.5}$$

with $\Omega \subset \mathbb{R}^N$ being a bounded Lipschitz domain, $a \in C^\alpha(\Omega)$, $a \geqslant 0$, a local minimizer $u \in W^{1,p}(\Omega)$ of $\mathcal{F}$, and a closed set $\Sigma \subset \Omega$ with

$$\dim_{\mathcal{H}}(\Sigma) > n-p-\varepsilon,$$

such that all points of Σ are non-Lebesgue points of (the precise representative of) u.

As will be clear from Subsection 6.5, and in particular from Theorem 6.6, the condition on the distance between p and q in the previous theorem cannot be relaxed. In the previous example the set of non-Lebesgue points of the minimizer is unrectifiable. This is the effect of the presence of x in the integrand, allowing to “distribute” the singularities of the minimizer on a Cantor type set. More comments can be found in Subsection 6.5.

After some sporadic come out in literature, see for instance [303] and related references, the study of regularity of minima of functionals with non-standard growth of (p,q)-type was initiated by Marcellini [235], [236], [237], [238], [239], who first identified a condition that, under suitable smoothness assumptions on the integrand F, ensures the regularity of minima. When referring to (p,q)-growth conditions (5.2), let me call the quantity $q/p > 1$ *the gap ratio of the integrand* F, or simply, *the gap*. Marcellini's approach prescribes that

$$\text{“the gap } \frac{q}{p} \text{ cannot differ too much from 1”}, \tag{6.1}$$

in other words, the numbers q and p cannot be too far apart. This approach is of course suggested by the counterexamples we have seen in Section 5, and in particular by (5.3) and (5.4). The application of (6.1) will be a standard in the next theorems, and the choice of the bound to assume on the gap q/p will change accordingly to the specific situation.

6.1. Lipschitz regularity and the gap

In order to give a first instance of the effect of assuming (6.1), I will present two sample theorems, which are not the most general ones available in literature, but which are nevertheless suitable to give a correct kind of flavor of the matter. I shall start considering simpler, autonomous functionals of the type

$$\mathcal{F}_s(v) := \int_\Omega F(Dv)\,\mathrm{d}x, \tag{6.2}$$

with $F(z)$ satisfying the following “(p,q)-version” of assumptions (2.16):

$$\begin{cases} z \mapsto F(z) \ \text{ is } \ C^2, \\ \nu|z|^p \leqslant F(z) \leqslant L(1+|z|^q), \\ \nu(1+|z|^2)^{(p-2)/2}|\lambda|^2 \leqslant \langle F_{zz}(z)\lambda,\lambda\rangle \leqslant L(1+|z|^2)^{(q-2)/2}|\lambda|^2. \end{cases} \tag{6.3}$$

Note that at this point the convexity assumptions $(6.3)_3$ are formulated according to the growth conditions in $(6.3)_2$. Unless otherwise specified, I am dealing with the general vectorial case $u\colon \Omega \to \mathbb{R}^N$, $N \geqslant 1$. The following scalar result of Marcellini is taken from [236]:

Theorem 6.1. *Let $u \in W^{1,q}_{\text{loc}}(\Omega)$ be a local minimizer of the functional $\mathcal{F}_s$ under the assumptions* (6.3), *in the scalar case $N = 1$; moreover, assume that*

$$\frac{q}{p} < \frac{n}{n-2} \quad \textit{when } n > 2. \tag{6.4}$$

Then $Du \in L^{\infty}_{\text{loc}}(\Omega, \mathbb{R}^n)$.

Here we see that assuming (6.1) in the form of (6.4) allows to get local boundedness of the gradient. In most cases *this is the focal point of regularity* for functionals with (p, q)-growth. Indeed, once this kind of result is achieved, for simpler functionals of the type (6.2) the higher regularity problem can be dealt with as for functionals with standard growth conditions (2.11). Roughly speaking, unless we are dealing with degenerate problems, the behavior of a non-standard growth functional differs from that of the standard ones only by the growth conditions in the gradient variable z, and therefore for large values of z. So, when already knowing that the minimizer u has a bounded gradient, the behavior at infinity of the function F becomes irrelevant, and the standard, higher regularity theory applies. This argumentation can be of course made rigorous, see for instance [238], [251].

Theorem 6.1 cannot be extended to the vectorial case $N > 1$, as is clear from the counterexamples valid already when $p = q$, see Section 3. For Theorem 6.1 one assumes that minimizers are a priori in $u \in W^{1,q}_{\text{loc}}(\Omega)$, while in general we have seen that they are a priori only in $u \in W^{1,p}_{\text{loc}}(\Omega)$, compare the previous section. *Getting rid of this integrability gap is actually the first step when proving regularity of minimizers under (p, q)-growth conditions*: passing from $u \in W^{1,p}_{\text{loc}}$ to $u \in W^{1,q}_{\text{loc}}$. This is crucial when proving higher regularity such as Lipschitz continuity, since by the right-hand side inequality $(6.3)_2$, the integral of $|Du|^q$ appears everywhere in the estimates. The following result is due to Esposito & Leonetti & myself, and is a particular case of the ones in [109]:

Theorem 6.2. *Let $u \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}_s$ under the assumptions* (6.3)*; moreover, assume that*

$$\frac{q}{p} < \frac{n+2}{n}. \tag{6.5}$$

Then $u \in W^{1,q}_{\text{loc}}(\Omega, \mathbb{R}^N)$.

Combining Theorems 6.1 and 6.2 we obtain

Theorem 6.3. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional $\mathcal{F}_s$ under the assumptions* (6.3), *in the scalar case $N = 1$; moreover, assume that the gap q/p satisfies* (6.5). *Then $Du \in L^{\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$.*

For a discussion on the borderline case $q = p(n+2)/n$ see also [103]. Further extensions are in [110]. Assuming the "Uhlenbeck structure", see Subsection 4.7, has effects also in the case of functionals with non-standard growth, see [238] and again [110] with [251]. Indeed we have

Theorem 6.4. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}_s$ under the assumptions* (6.3)*; moreover, assume that the gap q/p satisfies* (6.5), *and that the integrand F can be written as $F(z) \equiv g(|z|)$. Then $Du \in L^{\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN})$.*

The above results concern either higher integrability of Du, or its Lipschitz continuity. In the vectorial case a partial regularity theory is also available, though being not yet as complete as one would hope. I refer to the papers [138], [113], [38], [39], [40] for partial regularity results under (p, q)-growth conditions.

6.2. On the gap condition (6.1)

The reader will immediately notice that the bound in (6.4) is larger than the one in (6.5). Actually, except in a few special cases, it is not known in general if it is possible to assume (6.4) in Theorem 6.2, or what is the best bound for the gap q/p one has to assume when applying the principle in (6.1) to autonomous functionals of the type (6.2): *this is the main open problem in the theory.* Anyway, observe that both the bound in (6.2) and the one in (6.5) are smaller than the one allowing for the counterexample to regularity in (5.3). The situation changes when considering non-autonomous functionals of the type (5.1), for which the best bound for q/p is known, compare Subsection 6.5 below.

Before going on, I shall try to give a very rough explanation of why (6.1) comes up naturally as a condition ensuring regularity. Apart from those obviously given by the counterexamples of Section 5, there are indeed also technical reasons for considering (6.1) as a natural assumption. We have seen in Subsection 2.1 that the regularity of solutions to linear elliptic equations as in (2.4) strongly depends on how large the ellipticity ratio L/ν is; compare for instance (2.6) or the singular sets estimates of Theorems 4.5–4.6. Now, modulo a suitable approximation argument, Theorems 6.1–6.4 are proved making use of the Euler-Lagrange equation (in the scalar case, to which I restrict for the following discussion) of the functional $\mathcal{F}$:

$$\int_{\Omega} \sum_{i=1}^{n} F_{z_i}(Du) D_i \varphi \, \mathrm{d}x = 0, \quad \forall\, \varphi \in C^{\infty}_c(\Omega).$$

The following argumentation will be now purely formal, but it can be raised to the correct standard of rigor. In the previous equation let me take $D_s\varphi$ instead of φ; then let me integrate by parts

$$\int_\Omega \sum_{i,j=1}^{n} F_{z_i z_j}(Du) D_j(D_s u) D_i\varphi \, \mathrm{d}x = 0.$$

Therefore, letting $a_{i,j} := F_{z_i z_j}(Du(x))$, the function $w := D_s u$ satisfies the linear elliptic equation with measurable coefficients

$$\int_\Omega a_{i,j}(x) D_j w D_i\varphi \, \mathrm{d}x = 0, \quad \forall\, \varphi \in C_c^\infty(\Omega), \tag{6.6}$$

which is of the type (2.4), but this time the coefficients are elliptic but not bounded

$$|a_{i,j}(x)| \leqslant L(1+|Du(x)|^2)^{(q-2)/2}, \quad a_{i,j}(x)\lambda_i\lambda_j \geqslant \nu(1+|Du(x)|^2)^{(p-2)/2}|\lambda|^2. \tag{6.7}$$

The ellipticity ratio of the matrix $\{a_{i,j}(x)\}$, that is the ratio between the largest and the lowest eigenvalue of $\{a_{i,j}(x)\}$, can be this time bounded only by

$$R(Du) := L/\nu(1+|Du(x)|^2)^{(q-p)/2}, \tag{6.8}$$

which blows up when $|Du| \to \infty$. This tells us that (6.6) is an instance of a *non-uniformly elliptic equation*, an equation where the ratio between the largest and the lowest eigenvalue is not assumed to be a-priori bounded, and it actually depends on the solution itself. We have therefore a very critical situation: we would like to prove that the gradient is bounded, but on the other hand the ellipticity ratio, which is the quantity controlling the regularity of solutions, blows up exactly when $|Du| \to \infty$. At this stage the role of assuming a condition like (6.1) becomes clear: it serves to give a bound to the *rate of possible blow-up* of $R(Du)$. In other words, (6.1) controls the rate of non-uniform ellipticity of equation (6.6): if $R(Du)$ does not potentially blow up very fast in terms of Du, and this happens when q/p is not very large, then it actually stays bounded, otherwise it really blows up! A precursor of gradient bounds for the general non-uniformly elliptic equation is Simon, in his beautiful paper [281]. Here, Simon's conditions are a sort of re-formulation of the principle (6.1). Again, on non-uniformly elliptic equations see [214], [298], [280].

Anyway there are cases where condition (6.1) can be assumed in a form weaker than the ones considered up to now. In particular, we have seen that Theorems 6.1–6.4 require that $q/p \to 1$ when $n \to \infty$. By looking at the counterexamples in Subsection 5.1, we see that if in general $q/p \nrightarrow 1$, then minimizers become unbounded.

Now, it happens that when minimizers are bounded, or their boundary data is assumed to be bounded and the functional allows for a maximum principle, then we do not have to require that $q/p \to 1$ when $n \to \infty$ in order to prove regularity results for the gradient. We can just assume, for instance, $q < p + 2$; this is essentially the strategy introduced in [108], and then developed in the papers [37], [39]. See also the interesting maximum principle papers [145], [216].

6.3. Additional structures

Concerning functionals with (p, q)-growth conditions, there has been large literature over the last fifteen years. Let me emphasize a few main directions. The first concerns the so called anisotropic functionals, whose model is given in Section 5 by the functional $\mathcal{F}_2$. In this case the functional satisfies an additional structure/growth condition of the type

$$\sum_{i=1}^{n} |z_i|^{p_i} \leqslant F(z) \leqslant L\left(1 + \sum_{i=1}^{n} |z_i|^{p_i}\right). \tag{6.9}$$

The notation here is as for the functional $\mathcal{F}_2$ in the previous section. The essence is that every directional derivative is penalized with its own exponent; these functionals are naturally defined in the so called anisotropic Sobolev spaces [4], and more precise results can be obtained thanks to the peculiar structure coming into the play and yielding more information. Papers dedicated to the issue are, among others [303], [235], [147], [149], [35], [217], [218], [4], [43], [289], [225], [63], [226]. Assuming an additional structure as in (6.9), that is assuming that the integrand $F(z)$ is bounded from above and below by the same quantity, leads to better results in other cases. A relevant one is that of functionals naturally defined in Orlicz spaces, that is when we have growth and coercivity conditions of the type

$$\Phi(|z|) \leqslant F(x, v, z) \leqslant L(1 + \Phi(|z|)). \tag{6.10}$$

Here $\Phi\colon [0, \infty] \to [0, \infty]$ is a Young function, i.e. a convex, increasing function such that $\Phi(0) = 0$; see [271] for a comprehensive introduction to Orlicz spaces and more information on Young functions. If in turn $\Phi(t)$ satisfies $\nu t^p - L \leqslant \Phi(t) \leqslant L(1 + t^q)$, then the function $F(z)$ also satisfies (p, q)-growth conditions. There is large literature dedicated to functionals satisfying (6.10). In many of these papers the crucial assumption on the function Φ is the so called Δ_2-condition

$$\Phi(2t) \leqslant c\Phi(t) \tag{6.11}$$

that serves to exclude fast growth instances such as $\Phi(t) \equiv \exp(t^2)$. Moreover, another condition, namely the ∇_2-condition is also imposed, see [74], a condition dual to the one in (6.11): it serves to exclude slow growth instances such as $\Phi(t) \equiv t\log(1+t^2)$. To such cases I will nevertheless turn back in the next subsection. The basic, common approach in the papers dedicated to the structure (6.10) is to reproduce the results valid for functionals satisfying (2.11), viewing the case $\Phi(t) = t^p$ as a special one of (6.10). In a certain sense the function $F(z)$ is now "re-balanced" by assuming (6.10). Papers dedicated to the issue are, among others, [34], [294], [148], [62], [257], [146], [244], [245], [74]. I want to mention that an interesting bridge between (6.9) and (6.10) has been built by Cianchi [63], while a very complete picture concerning also certain classes of elliptic equations in divergence form is given by Lieberman [223], relying on the techniques in [281].

6.4. Extreme cases

Considering (p,q)-growth conditions turns out to be still too restrictive when dealing with certain classes of variational integrals. Here are two examples of functionals not satisfying (p,q)-growth conditions, and for opposite reasons:

$$\mathcal{F}_7(u) = \int_\Omega |Du|\log(1+|Du|)\,\mathrm{d}x, \quad \mathcal{F}_8(u) := \int_\Omega \exp(|Du|^2)\,\mathrm{d}x.$$

The former does not meet (p,q)-growth conditions because the integrand grows too slowly in the gradient variable, and it fails to be polynomially super-linear; the latter because the integrand grows faster than any power. Nevertheless, for both of the functionals a fully satisfying regularity theory is available, even in the vectorial case $N \geqslant 1$ (this is also due to the "Uhlenbeck structure" shared by both the integrands, see Subsection 4.7).

As for $\mathcal{F}_8$, the C^∞-nature of minimizers was shown by Lieberman [224], who relied very much on the peculiar structure of the integrand. A much more general theory is offered by Marcellini [238], [239], who is able to treat, also in the case $N \geqslant 1$, a very wide class of variational integrals with fast growth in the gradient, including any finite composition of exponentials, i.e. functionals of the form

$$\int_\Omega \exp(\exp(\ldots\exp(|Du|^2)\ldots))\,\mathrm{d}x. \tag{6.12}$$

Extensions to a class of non-autonomous integrals can be found in [243].

As for $\mathcal{F}_7$, the continuity of the gradient was obtained in two dimensions $n=2$ by Frehse & Seregin [133] and Fuchs & Seregin [138], who explicitly raised the problem of proving the result in higher dimensions. This was settled by Siepe & myself in [251], where the proof of the Hölder continuity of the gradient of minima in any dimension $n>2$ is achieved in the general vectorial case $N \geqslant 1$. The proof in [251] relies on the simple observation that, no matter how slowly the integrand $F(z) = |z|\log(1+|z|)$

grows, when looking at the the second derivative matrix F_{zz}, it does not decay fast enough yet to allow for irregularity. In other words, there is ellipticity enough to regularize minimizers. The same observation allows to treat more general integrals with slower growth, and for instance any finite composition of logarithms is allowed, i.e. functionals growing like

$$\int_{\Omega} |Du| \log(1 + \log(\dots \log(1 + |Du|^2) \dots)) \,\mathrm{d}x. \tag{6.13}$$

This one is in some sense "dual" to that in (6.12), in the same way as $\mathcal{F}_7$ and $\mathcal{F}_8$ can be considered dual to each other (there is a way to make this rigorous, using the theory of duality in Orlicz spaces, see [271]). Regularity for minima of functionals as in (6.13) has been obtained by Fuchs & myself in [136]. Further developments can be found in [36], [241].

6.5. Non-autonomous functionals

Up to now I have confined myself to simple, autonomous integrands of the type (6.2). Now I am going to deal with more general, non-autonomous ones of the type

$$\mathcal{F}_{na}(v) := \int_{\Omega} F(x, Dv) \,\mathrm{d}x, \tag{6.14}$$

still satisfying non-standard growth conditions of (p, q)-type (5.2). If we look at the case $p = q$, and especially at Theorem 2.3 and at Subsection 2.2, we see that the precise degree of regularity of the integrand $F(x, z)$ with respect to the x-variable is irrelevant in order to get the Hölder continuity of minimizers. Moreover, also when looking at Theorem 2.5, we see that the degree of Hölder continuity of $F(x, z)$ with respect to x only influences the degree of Hölder continuity of Du, but not the fact that Du is Hölder continuous or not; in other words, any degree α of Hölder continuity of $x \to F(x, z)$ suffices in order to get a Hölder continuous gradient. The modest influence of the presence of the x-variable in the integrand is also clear when looking at the techniques of proof of Theorems 2.4 and 2.5, see [68], [158], [231], where the presence of x is treated essentially using local perturbation methods. When dealing with (p, q)-growth conditions *the situation drastically changes*, and the novelty is that the presence of x cannot be treated as a perturbation anymore. This can be guessed by looking at the structure of the integrands in functionals $\mathcal{F}_3$, $\mathcal{F}_4$ in Section 5, that I will call $F_3(x, z)$ and $F_4(x, z)$, respectively. In both cases, if we keep x fixed and let z vary, the integrand satisfies standard growth conditions; for instance $z \to F_3(x, z)$ has p-growth if $a(x) = 0$, and it has q-growth if $a(x)$, while it globally satisfies (p, q)-growth conditions *since the variable x is varying* *simultaneously with* z. A similar argumentation works of course for the integrand of $\mathcal{F}_{3.2}$ in Subsection 5.2, and for $F_4(x,z)$ of $\mathcal{F}_4$, with which the next section is concerned. This immediately tells us that in the case of functionals with (p,q)-growth, the effect of x can be very relevant, since it is itself responsible for the (p,q)-growth conditions to appear! Indeed, as demonstrated in the papers [111], [126], when dealing with functionals of the type (6.14), the regularity of minima is ruled by a subtle interaction between the regularity of the function $x \to F(x,z)$, and the size of the gap q/p. The counterexample of Subsection 5.2 already tells us that, in order to create singularities, the numbers q and p must be far from each other accordingly to the size of α. This is a general phenomenon, that reveals to be another instance of principle (6.1): for functionals of the type (6.14), the condition allowing to prove regularity results for minimizers is

$$\frac{q}{p} < \frac{n+\alpha}{n}, \quad \alpha \in (0,1], \tag{6.15}$$

where this time α is the Hölder continuity exponent of $x \to F(x,z)/(1+|z|^q)$, compare $(6.19)_4$ below. Therefore: *the less regular with respect to* x *the integrand* $F(x,z)$ *is, the less we are allowed to get* q/p *far from* 1. Condition (6.15) is actually sharp, as the counterexample from Subsection 5.2 immediately shows. For instance, let me report the following result, taken from [111]:

Theorem 6.5. *Let* $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ *be a local minimizer of the functional* $\mathcal{F}_3$ *and assume that* $0 \leqslant a \in C^{0,\alpha}(\Omega)$ *with* (6.15). *Then* $Du \in W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^{nN})$.

There is even a more precise version, arising when minimizers are bounded: in this case condition (6.15) can be replaced by $q \leqslant p+\alpha$, which is again the one appearing in Theorem 5.1. This is according to the discussion at the end of Subsection 6.2; see [103] for more details, and also for the borderline cases $q = p(n+\alpha)/n$ and $p = q+\alpha$, when Theorem 6.5 is still valid. (I hope this paper will be ready soon!)

Let me notice that there is clearly a gap between the autonomous condition (6.5) and the non-autonomous one (6.15). Even in the most favorable case $\alpha = 1$ the two conditions do not coincide, and (6.15) is more restrictive than (6.5), this being a non-fixable effect of the presence of x. On the other hand, a pleasant consequence of Theorem 6.5 and of Theorem 6.6 below is that while for autonomous functionals of the type (3.6) it is not known what is in general the best bound to assume on the gap q/p, compare Subsection 6.2, here we see that in the non-autonomous case the best possible bound is (6.15). This is again a consequence of the counterexample in Subsection 5.2.

Theorem 6.6 is a particular case of a more general theory, whose beginnings are settled down in [111], and which I am now going to outline. This goes via the analysis of the so-called *Lavrentiev Phenomenon* (LP), which the functionals of the type in (6.14) typically exhibit when under (p,q)-growth conditions. Roughly speaking, LP occurs for a map $v \in W^{1,p}(\Omega, \mathbb{R}^N)$, when it is not possible to find a sequence of more regular maps $v_n \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ such that $v_n \rightharpoonup v$ in $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, and the following "approximation in energy" takes place:

$$\int_A F(x, Dv_n)\,\mathrm{d}x \to \int_A F(x, Dv)\,\mathrm{d}x \tag{6.16}$$

for every $A \subset\subset \Omega$, with A being an open subset. This is actually a re-adaptation of the original definition that perfectly fits here, and that I am adopting from now on for the sake of simplicity. When the Lavrentiev Phenomenon occurs for a local minimizer u then it follows, in particular, that it is not possible to realize locally minimizing sequences $\{u_n\}_n$ for $\mathcal{F}$ with more regular maps $u_n \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. LP is a clear obstruction to minimality, since if u is a minimizer such that $u \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, then by the very definition there is not LP for u. It is interesting and significant to see that $\mathcal{F}$ never exhibits LP either when $p = q$ or when $F(x,z) \equiv F(z)$, see [111], so that LP results from the coupling of the (p,q)-growth conditions with the dependence on x in the integrand. For a nice survey on LP, I recommend [48], while, in the setting of functionals with (p,q)-growth, fundamental contributions are due to Zhikov [308], [309], where several examples of LP are given. A striking example of LP for functionals with variable growth exponent (see the last section) has been offered by Foss [127]; LP also play an important role in non-linear elasticity, see [25], [130], [131]. A useful way to quantify LP can be introduced according to Buttazzo & Mizel [49], as I will briefly explain now. From now I shall consider integrands $F(x,z)$ which are convex with respect to z, in order to gain lower semicontinuity for the related functionals; a more general situation can be found again in [49], [111]. Following [49], let me define the relaxed functional

$$\overline{\mathcal{F}}(v, B_r) := \inf_{v_n}\Big\{\liminf_n \int_{B_r} F(x, Dv_n)\,\mathrm{d}x\colon\ v_n \in W^{1,q}(B_r, \mathbb{R}^N),\quad v_n \rightharpoonup v \ \text{ in } \ W^{1,p}(B_r, \mathbb{R}^N)\Big\}, \tag{6.17}$$

where $B_r \subset\subset \Omega$ is a ball with radius $r > 0$. As mentioned above, since $F(x,z)$ is convex with respect to z, we have

$$\mathcal{F}_{na}(v, B_r) := \int_{B_r} F(x, Dv)\,\mathrm{d}x \leqslant \liminf_n \int_{B_r} F(x, Dv_n)\,\mathrm{d}x$$

whenever $v_n \rightharpoonup v$ and $v_n \in W^{1,p}(B_r, \mathbb{R}^N)$. Therefore $\mathcal{F}_{na}(v, B_r) \leqslant \overline{\mathcal{F}}(v, B_r)$ for every $v \in W^{1,p}(B_r, \mathbb{R}^N)$, and it is possible to define the following, non-negative

Lavrentiev Gap Functional:

$$\mathcal{G}(v, B_r) := \overline{\mathcal{F}}(v, B_r) - \mathcal{F}_{na}(v, B_r) \geqslant 0, \quad \forall\, v \in W^{1,p}(B_r, \mathbb{R}^N).$$

The value of the functional $\mathcal{G}(v, B_r)$ gives a measure of the impossibility of approximating in energy, that is (6.16), of the map v by a sequence of more regular maps. Indeed (6.16) occurs with $A \equiv B_r$ if and only if $\mathcal{G}(v, B_r) = 0$. Now, let $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of $\mathcal{F}_{na}$. Then, clearly,

$$u \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \Longrightarrow \mathcal{G}(u, B_r) = 0, \quad \forall\, B_r \subset\subset \Omega. \tag{6.18}$$

This leads to the following developments: usually one proves regularity of minimizers to exclude the LP, that is (6.18); in [111] we followed the reverse procedure, using the absence of LP to prove regularity. Let me consider the following assumptions, parallel to those in (2.16):

$$\begin{cases} z \mapsto F(x,z) \text{ is } C^2, \\ \nu |z|^p \leqslant F(x,z) \leqslant L(1+|z|^q), \\ \nu |z|^{p-2}|\lambda|^2 \leqslant \langle F_{zz}(x,z)\lambda, \lambda\rangle \leqslant L(1+|z|^2)^{(q-2)/2}|\lambda|^2, \\ |F(x,z) - F(y,z)| \leqslant L|x-y|^\alpha (1+|z|^q). \end{cases} \tag{6.19}$$

Then we have the following result from [111], with a few improvements from [103]:

Theorem 6.6. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}_{na}$ under the assumptions* (6.19). *Assume also that* (6.15) *holds, together with*

$$\mathcal{G}(u, B_r) = 0. \tag{6.20}$$

Then $Du \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN})$.

Condition (6.20) is not tautological, and it is necessary by (6.18), but it may appear difficult to verify in general. Nevertheless, a twist happens now: for all model cases in literature, and in particular for the functionals $\mathcal{F}_1$–$\mathcal{F}_6$, we have

$$(6.15) \Longrightarrow (6.20), \tag{6.21}$$

establishing a deeper connection between the regularity theory and LP. Therefore, for all the model cases known, and in particular for $\mathcal{F}_1$–$\mathcal{F}_6$ from Section 5, assumptions (6.19) and (6.15) suffice for the basic $W^{1,q}$ regularity of minima. This path is again followed in [111], building on a previous work of Zhikov, where large classes for functionals for which (6.21) occurs have been demonstrated. Further results on LP can be found in [250]. *It remains an open problem to establish the validity of* (6.21) *under the only assumptions* (6.19). Interesting developments to [111] can be found in [30], [40], [57], [69].

7. Variable growth exponents

We have seen that in the case of functionals with (p,q)-growth of the type (5.1), the regularity theory for minima is still far from finding a definitive and general setting, being linked, for instance, to the analysis of Lavrentiev Phenomenon. I have to say that my feeling is that, in the spirit of the classical Calculus of Variations, regularity results should be chased by looking at special classes of functionals and thinking of relevant model examples, thereby limiting the degree of generality one wants to achieve. One of such relevant classes, for which a general and rather complete theory is now available, is with no doubt the one of functionals with the so called $p(x)$-growth, i.e. functionals of the type (2.1) satisfying the following "variable exponent version" of the growth conditions in (2.11):

$$\nu|z|^{p(x)} \leqslant F(x,y,z) \leqslant L(1+|z|^{p(x)}). \tag{7.1}$$

The exponent function $p\colon \Omega \to (1,\infty)$ will be here considered to be continuous and satisfy

$$1 < \gamma_1 \leqslant p(x) \leqslant \gamma_2 < \infty. \tag{7.2}$$

The clear prototype is the functional $\mathcal{D}_{p(x)}$ we already met in Section 6:

$$\mathcal{D}_{p(x)}(v) := \int_\Omega |Dv|^{p(x)}\,\mathrm{d}x.$$

The assumption (7.2) clearly serves to ensure that $\mathcal{D}_{p(x)}$ keeps far from the total variation functional [287] as well as from the so called ∞-Laplacean [20]. This energy shows up when considering a number of models from Mathematical Physics: homogenization of strongly anisotropic material, as pioneered by Zhikov [307], [311], electro-rheological fluids as modelled by Rajagopal & Růžička [270], [274], temperature dependent viscosity fluids, as again conceived by Zhikov [310], image processing models by Chen & Levine & Rao as in [58]. More generally, a functional as $\mathcal{D}_{p(x)}$ serves when modelling physical situation with strong anisotropicity, the anisotropic nature of the situation being described by the appearance of the x-variable in the growth exponent. Here I will confine myself to report the basic regularity results for minimizers of the functional $\mathcal{D}_{p(x)}$ available in literature up to now. The same will apply to solutions to the related Euler-Lagrange system

$$\operatorname{div}(p(x)|Du|^{p(x)-2}Du) = 0. \tag{7.3}$$

The results I am going to present are also valid for more general functionals, equations and systems with "$p(x)x$-growth", provided suitable distinctions between the scalar and vectorial case are done; for this I refer to [5], [6], [8], [66]. I emphasize here that, when referring to $\mathcal{D}_{p(x)}$ and the system (7.3), I shall consider the problem in the general vectorial case $u\colon\ \Omega \to \mathbb{R}^N$ and $N \geqslant 1$. We have seen in the previous section that, when considering non-autonomous functionals with (p,q)-growth conditions, the regularity of minimizers depends on a subtle interaction between the gap q/p and the regularity of the integrand $F(x,z)$ with respect to the x-variable. In particular, condition (6.15) tells us that, in a certain sense, the Hölder continuity with respect to x serves to "re-balance" the distance between p and q created by the very fact that x varies, compare Subsection 6.5. In the case of the functional $\mathcal{D}_{p(x)}$ the key idea is to think that on small domains, say small balls $B_s \subset \Omega$, the gap q/p of the integrand $|z|^{p(x)}$ can be made arbitrarily near 1 since

$$q := \sup_{B_s} p(x), \quad p := \inf_{B_s} p(x),$$

and the function $p(x)$ is continuous. Therefore, since proving local regularity results is something that can be done by reducing the problem to an arbitrarily small open subset of Ω, and then concluding with a standard covering argument, we see at once that for the functional $\mathcal{D}_{p(x)}$, if thinking of condition (6.15), any, possibly small, Hölder exponent $\alpha > 0$ suffices in order to get regularity of minima in the sense of Theorem 6.6. We have actually much more: due to the peculiar structure of $\mathcal{D}_{p(x)}$, condition (6.15) admits a borderline case, that is the so called log-continuity assumption, first introduced by Zhikov [309] to treat the Lavrentiev Phenomenon related to $\mathcal{D}_{p(x)}$. This goes as follows: let me denote by $\omega(\cdot)$ the modulus of continuity of the exponent function $p(x)$, that is

$$\omega(s) := \sup_{\substack{B_s \Subset \Omega \\ x,y \in B_s}} |p(x) - p(y)|.$$

Then the log-continuity assumption prescribes that

$$\limsup_{s\to 0} \omega(s) \log \frac{1}{s} = L < \infty. \tag{7.4}$$

Such an assumption turns out to be crucial: Zhikov proved that the failure of (7.4) is a possible cause of discontinuities of minima [309], see also [180]. On the other hand, assuming (7.4) allows to prove higher integrability of minimizers, that is, with $u \in W^{1,1}(\Omega, \mathbb{R}^N)$ being a local minimizer of $\mathcal{D}_{p(x)}$, there exists a $\delta > 0$ such that

$$\int_\Omega |Du|^{p(x)(1+\delta)}\,\mathrm{d}x < \infty. \tag{7.5}$$

Moreover, Zhikov proved that (7.4) is a sort of "universal condition", linking regularity of solutions and minima to the structure of the spaces $L^{p(x)}(\Omega, \mathbb{R}^N)$ and to Lavrentiev Phenomenon for the functional $\mathcal{D}_{p(x)}$. Let us have a rapid look at the situation. The space $L^{p(x)}(\Omega, \mathbb{R}^N)$ is defined as

$$L^{p(x)}(\Omega, \mathbb{R}^N) := \left\{ v\colon\ \Omega \to \mathbb{R}^N :\ v \text{ is measurable and } \int_\Omega |v|^{p(x)x}\,\mathrm{d}x < \infty \right\}, \tag{7.6}$$

and once equipped with the Luxemburg norm

$$\|v\|_{L^{p(x)}(\Omega,\mathbb{R}^N)} := \inf\left\{ \lambda > 0\colon\ \int_\Omega \left|\frac{v}{\lambda}\right|^{p(x)} \mathrm{d}x \leqslant 1 \right\}$$

it becomes a Banach space. The space $L^{p(x)}(\Omega, \mathbb{R}^N)$ is an instance, actually the most important and popular one, of the so called Orlicz-Musielak spaces [263], [85]. Accordingly, the *generalized* $W^{1,p(x)}(\Omega, \mathbb{R}^N)$ *space* is defined by

$$W^{1,p(x)}(\Omega, \mathbb{R}^N) := \{ v \in L^{p(x)}(\Omega, \mathbb{R}^N)\colon\ Dv \in L^{p(x)}(\Omega, \mathbb{R}^{nN}) \},$$

where Dv obviously denotes the distributional gradient of the map v. This also becomes a Banach space with the norm defined by

$$\|v\|_{W^{1,p(x)}(\Omega,\mathbb{R}^N)} := \|v\|_{L^{p(x)}(\Omega,\mathbb{R}^N)} + \|Dv\|_{L^{p(x)}(\Omega,\mathbb{R}^{nN})}.$$

Zhikov essentially proved that (7.4) implies the absence of Lavrentiev Phenomenon for the functional $\mathcal{D}_{p(x)}$, due to the approximation property in (6.16), and he also proved that the convolution/mollification operator is bounded when assuming condition (7.4). Subsequently, a massive quantity of interesting contributions have been given on the spaces $L^{p(x)}(\Omega, \mathbb{R}^N)$; there is no room here to give account of this, and I will refer to the recent excellent surveys [86], [276]. I hereby just want to mention the results obtained by Diening & Růžička [87], who proved that singular integral operators are bounded in $L^{p(x)}$ if and only if (7.4) is satisfied, while boundedness results in $L^{p(x)}$ for fractional maximal-type operators have been obtained by Kokilashvili & Samko [196]; see also [85] for more on Harmonic Analysis in $L^{p(x)}$-spaces.

In a series of papers, Acerbi, Coscia and myself, investigated the regularity properties of local minimizers of $\mathcal{D}_{p(x)}$ when assuming condition (7.4) and/or suitable reinforcements [5], [6], [7], [8], [66]. Further contributions to regularity, also for non-variational situations and in the parabolic case, are those in [11], [118], [119], [59], [10], [17], [15], [16]. I am starting with the Hölder continuity of minimizers.

Theorem 7.1. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{D}_{p(x)}$ under the assumption*

$$\limsup_{s \to 0} \omega(s) \log \frac{1}{s} = 0. \tag{7.7}$$

Then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ for every $\alpha < 1$.

The proof of this result in the scalar case is contained in [5]; the one in the vectorial case goes along the lines of the scalar one, taking into account the estimates in [66]. It is to be noted that assuming only (7.4) in the scalar case $N = 1$, one can prove that $u \in C^{0,\beta}_{\mathrm{loc}}(\Omega)$ for some (small) $\beta > 0$, see [11]. This is not by chance; indeed by [5] we still infer that for every $\tilde{\alpha} \in (0,1)$ there exists $\varepsilon \equiv \varepsilon(\tilde{\alpha}) > 0$ such that if

$$\limsup_{s \to 0} \omega(s) \log \frac{1}{s} \leqslant \varepsilon,$$

then $u \in C^{0,\tilde{\alpha}}_{\mathrm{loc}}(\Omega)$. In other words, controlling the oscillations of the exponent function $p(x)$ allows to control the degree of regularity of local minimizers.

In order to get the Hölder continuity of the gradient it is unavoidable to assume that $p(x)$ is itself Hölder continuous, that is

$$\omega(s) \leqslant L s^{\alpha}, \quad \alpha > 0, \tag{7.8}$$

which is obviously stronger that (7.7). The result, taken from [66], is the following

Theorem 7.2. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{D}_{p(x)}$ under the assumption* (7.8). *Then $Du \in C^{0,\beta}_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN})$ for some $\beta < \alpha$.*

This result is sharp in the sense that if $p(x)$ is not Hölder continuous then the gradient is not even continuous in general, as shown in [179]. Moreover, it is not possible to get that $Du \in C^{0,\beta}_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN})$ for every $\beta < 1$, the counterexample already working in the case where $p(x)$ is a constant function [302].

The proof of Theorems 7.1–7.2 is based on a delicate combination of ingredients: a careful localization argument starting from the higher integrability property (7.5); a perturbation-in-the-exponent method, build on a combined use of reverse Hölder inequalities and estimates in the space $L \log L$. The regularity of local minima of $\mathcal{D}_{p(x)}$

is indeed obtained by comparison with minimizers of (2.18), for a suitable choice of the fixed exponent p. The variations in the exponent naturally make quantities as

$$\int |Du|^{p(x)(1+\delta)} \log(\mathrm{e} + |Du|)\,\mathrm{d}x$$

appear, and these have to be estimated very carefully in order to get the result under the optimal assumption (7.7). At this stage another crucial role is played by the so-called "stability of the estimates" for solutions to the p-Laplacean system (2.19): all constants involved in the local $C^{0,1}$ and $C^{1,\alpha}$ estimates of solutions to (2.19), including α, do not blow up or degenerate, as long as p varies in a compact subset of $(1,\infty)$. For this I again recommend Manfredi's thesis [232] or [122]. The local regularity results for minimizers of Theorems 7.1–7.2 immediately extend to solutions to the system (7.3), as I myself showed in the lectures of the Paseky course, the proof being actually simpler. The proof of the $C^{0,\alpha}$ regularity can be obtained as a corollary of the results in [8], while the proof of the $C^{1,\beta}$ result for general equations is unfortunately not explicitly written anywhere, but it easily follows from the arguments in [66], where on the other hand the model system (7.3) is obviously covered. Again, Theorems 7.1 and 7.2 extend to more general functionals with $p(x)$-growth, i.e. functionals of the type (2.1) whose integrand is in a suitable sense controlled by $|Du|^{p(x)}$; see also [5], [107].

In [8], Acerbi and myself have given yet another proof of Theorem 7.1 that eventually follows as a particular case of the next Calderón-Zygmund type result. Let me consider the non-homogeneous $p(x)$-Laplacean system

$$\operatorname{div}(p(x)|Du|^{p(x)-2}Du) = \operatorname{div}(|F|^{p(x)-2}F), \tag{7.9}$$

where $F\colon\ \Omega \to \mathbb{R}^{nN}$ is a prescribed vector field such that

$$\int_\Omega |F|^{p(x)}\,\mathrm{d}x < \infty.$$

By a weak solution to (7.9) I mean, of course, a map $u \in W^{1,p(x)}(\Omega,\mathbb{R}^N)$ such that

$$\int_\Omega p(x)|Du|^{p(x)-2}DuD\varphi\,\mathrm{d}x = \int_\Omega |F(x)|^{p(x)-2}F(x)D\varphi\,\mathrm{d}x$$

for every test function $\varphi \in W^{1,p(x)}(\Omega,\mathbb{R}^N)$ with compact support in Ω. For the related existence theory the reader should look at the work of Růžička [274]. We have

Theorem 7.3. *Let $u \in W^{1,p(x)}(\Omega, \mathbb{R}^N)$ be a weak solution to the non-homogeneous $p(x)$-Laplacean system* (7.9) *under the assumption* (7.7)*. Then*

$$|F|^{p(x)} \in L^q_{\text{loc}}(\Omega) \Longrightarrow |Du|^{p(x)} \in L^q_{\text{loc}}(\Omega), \quad \forall\, q > 1. \tag{7.10}$$

This result is again sharp [309], in the sense that without assuming at least (7.4) the statement is not true; for more precise comments see [8, Remark 2]. Theorem 7.1 follows by choosing $F \equiv 0$ and then applying the Sobolev embedding theorem. We are also able to provide a local a priori estimates for the gradient Du in terms of certain natural reverse Hölder inequalities, see Theorem 2 in [8]. In the case $p(x) \equiv$ constant, Theorem 7.3 is due to T. Iwaniec [189] in the scalar case $N = 1$ and to DiBenedetto & Manfredi [83] for the case $N > 1$; for L^q-estimates for the p-Laplacean operator see also the paper by Caffarelli & Peral [51]. The proof of Theorem 7.3 yields anyway new results already in this classical case, in that we are able to treat also a class of non linear degenerate elliptic equations with p-growth in divergence form. The methods in [8] readily extend to cover more general right-hand sides for the $p(x)$-Laplacean system, as for instance

$$\operatorname{div}(p(x)|Du|^{p(x)-2}Du) = \operatorname{div} F.$$

Let me conclude going back to the case of a fixed non-variable growth exponent p. Very recently, in [9], the elliptic results in [189] and [83] have been extended to a large class of parabolic operators whose model type is the non-homogeneous parabolic p-Laplacean operator

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = \operatorname{div}(|F|^{p-2}F) \tag{7.11}$$

under the assumption

$$p > \frac{2n}{n+2}. \tag{7.12}$$

In this situation the system (7.11) is considered in a cylindrical domain $C := \Omega \times [0, T)$, where $T > 0$ and $\Omega \subset \mathbb{R}^n$ is, as usual, a bounded domain, while the solution u is with no loss of generality considered in the space

$$u \in C^0((0,T); L^2(\Omega, \mathbb{R}^N)) \cap L^p(0,T; W^{1,p}(\Omega, \mathbb{R}^N))$$

and $N \geqslant 1$; finally $F \in L^p(C, \mathbb{R}^{nN})$. Under the previous assumptions Acerbi & myself proved the following analog of (7.10):

$$|F|^p \in L^q_{\text{loc}}(C) \Longrightarrow |Du|^p \in L^q_{\text{loc}}(C) \quad \forall\, q > 1. \tag{7.13}$$

Moreover, we were also able to provide precise local estimates of reverse-Hölder type, bounding the L^{pq} norm of Du in terms of that of F and the L^p norm of Du itself. Such estimates are natural in that, in the homogeneous case $F \equiv 0$, by letting $q \nearrow \infty$ we recover from them the classical local $C^{0,1}$-estimates of DiBenedetto and Friedman [82]. The lower bound in (7.12) is necessary in order to obtain the result in (7.13). The new technical contribution of [9] consists in providing a method which is completely free of Harmonic Analysis tools. Indeed in the papers [189], [83] crucial use is made of various maximal operators; this is not possible in the case of the systems as (7.11). Indeed all estimates must be carried out according to the "intrinsic geometry viewpoint" of DiBenedetto [81], and therefore on parabolic cylinders whose size depends on the solution itself. Such cylinders are a priori arbitrary, and therefore not related to any fixed maximal operator. On the contrary, we rely on a new method involving several diffferent ingredients. For instance, we are directly using certain Calderón-Zygmund type coverings of the level sets of the gradient Du, which are locally adapted to the solution, and use them in combination with the $C^{0,1}$ estimates available in the case of the homogeneous parabolic p-Laplacean system [82], that is (7.11) with $F \equiv 0$. Moreover, since we are not using any maximal type operator, we cannot use the so called "good-λ-inequality" principle as in [8]; on the contrary, we introduce an analog version of that, working again on Calderón-Zygmund cylinders directly: we called it the "large-M-inequality" principle. This time the method is flexible enough to include more general systems with possibly discontinuous coefficients of the type

$$u_t - \operatorname{div}(a(x,t)|Du|^{p-2}Du) = \operatorname{div}(|F|^{p-2}F),$$

where $\nu \leqslant a(x,t) \leqslant L$ may be discontinuous in a suitable VMO/BMO fashion; this extends previous, elliptic results of Kinnunen & Zhou [199], where again maximal operators are crucially employed. Studying Calderón-Zygmund type estimates for equations with discontinuous coefficients has been the object of intensive investigation at length: see [60], [88], [42], and references. Moreover, the method extends to *all degenerate/singular parabolic equations* in divergence form of the type

$$u_t - \operatorname{div} a(x,t,Du) = \operatorname{div}(|F|^{p-2}F),$$

where the vector field a satisfies the assumptions in (2.20), suitably recast for the case under consideration, but just requiring continuity dependence with respect to (x,t) and not Hölder continuity.

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