Proceedings of the Royal Society of Edinburgh, **120A**, 185–189, 1992

Rank-one convexity does not imply quasiconvexity

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(MS received 13 November 1991)

1. Introduction

We consider variational integrals

$$I(u)=\int_\Omega f(Du(x))\,dx, \tag{1}$$

defined for (sufficiently regular) functions $u\colon \Omega\to\mathbf{R}^m$. Here Ω is a bounded open subset of $\mathbf{R}^n$, $Du(x)$ denotes the gradient matrix of u at x and f is a continuous function on the space of all real $m\times n$ matrices $M^{m\times n}$. One of the important problems in the calculus of variations is to characterise the functions f for which the integral I is lower semicontinuous. In this connection, the following notions were introduced (see [3], [9], [10]).

(i) f is *rank-one convex* if for each matrix $A\in M^{m\times n}$ and each rank-one matrix $B\in M^{m\times n}$ the function $t\to f(A+tB)$ is convex.

(ii) f is *quasiconvex* if for any matrix $A\in M^{m\times n}$ and any smooth function $\varphi\colon\Omega\to\mathbf{R}^m$ compactly supported in Ω the inequality $\int_\Omega f(A+D\varphi)\,dx\geqq \int_\Omega f(A)\,dx$ holds true. The class of quasiconvex functions is independent of Ω (See [3], [10].)

(iii) f is *polyconvex* if $f(X)=$ convex function of minors of the matrix X.

It is well-known that (iii)$\Rightarrow$(ii)$\Rightarrow$(i). (See [10], [3].) The quasiconvexity of f is equivalent to the lower semicontinuity of I with respect to the w^*-convergence in $W^{1,\infty}$, a result due to Morrey. (See [9], [10]. See also [1].) This might seem a satisfactory answer to our problem, but it turns out that it can be rather difficult to decide whether or not a given function is quasiconvex. In fact there are many explicit examples of functions for which it is not known whether or not they are quasiconvex (See [2], [6], [7], [13], for example.) On the other hand, it is relatively easy (at least in principle) to decide whether or not a given function f is rank-one convex or polyconvex. It is therefore interesting to look at the implications (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii). An old result in this direction is that for quadratic forms (i)$\Rightarrow$(ii) is true (see [10], [3]). In [12] it is shown that rank-one convexity of f implies that the inequality in (ii) is satisfied for certain special classes of functions φ. There are examples of quadratic forms on $M^{3\times 3}$ showing that (ii) does not imply (iii). (See [16], [11].) On $M^{2\times 2}$ every quasiconvex quadratic form is polyconvex ([16], [11]). For general functions on $M^{2\times 2}$ (ii) does not imply (iii) ([2], [14]).

Here we show that (i) does not imply (ii) for $n \geqq 2$ and $m \geqq 3$. This is in accordance with Morrey's conjecture (made in 1952) that (i) does not imply (ii) for $n \geqq 2$ and $m \geqq 2$. (It seems he was not so sure about the conjecture in his 1966 book **[10]**.)

In **[4]** it is shown that if we allow f to be discontinuous and to take the value $+\infty$, then (i) does not imply (ii) for $n \geqq 3$, $m \geqq 3$ or $n \geqq 2$, $m \geqq 4$. As shown in the same paper, the idea of these examples does not work for functions which are everywhere finite.

As it was pointed out to me by Luc Tartar, the example given here is reminiscent of an example which he gave in connection with a theorem in compensated compactness. See **[15]**, pp. 185–6. Another related example is Example 3.5 from **[5]**.

2. Constructions

The following lemma will be useful. The lemma can be found for example in **[5]** or **[8]**. It also follows from the result of Morrey mentioned above. We sketch the proof for the convenience of the reader.

LEMMA 1. *A continuous function $f: M^{m\times n} \to \mathbf{R}$ is quasiconvex if and only if*

$$\int_{[0,1]^n} f(A + Du(x))\,dx \geqq f(A)$$

for each $A \in M^{m\times n}$ and each smooth function $u: \mathbf{R}^n \to \mathbf{R}^m$ which is periodic with respect to $\mathbf{Z}^n$, i.e. $u(x+l) = u(x)$ for each $x \in \mathbf{R}^n$ and each $l \in \mathbf{Z}^n$.

Proof. Let $f: M^{m\times n} \to \mathbf{R}$ be continuous and quasiconvex. Let $0 < \varepsilon < \frac{1}{2}$. Let $\eta_\varepsilon: [0,1]^n \to \mathbf{R}$ be a smooth function which is equal to 1 on $[\varepsilon, 1-\varepsilon]^n$, vanishes in a neighbourhood of the boundary of $[0,1]^n$ and satisfies $|D\eta_\varepsilon| \leqq 2/\varepsilon$. For a periodic function $u: \mathbf{R}^n \to \mathbf{R}^m$ we define $u_\varepsilon: B \to \mathbf{R}^m$ by

$$u_\varepsilon(x) = \varepsilon^2 \eta_\varepsilon(x) u(x/\varepsilon^2)$$

Since u_ε is smooth and compactly supported in $(0,1)^n$, the quasiconvexity of f implies

$$f(0) \leqq \int_{[0,1]^n} f(Du_\varepsilon(x))\,dx \tag{2}$$

for each $0 < \varepsilon < \frac{1}{2}$. It is easily seen that for $\varepsilon \to 0$ the limit of the integrals on the right-hand side of (2) is $\int_{[0,1]^n} f(Du(x))\,dx$. This proves the "only if" part of our statement. The "if" part is trivial. The proof is finished. □

Let us consider the function $w: \mathbf{R}^2 \to \mathbf{R}^3$ given by

$$w(x) = \frac{1}{2\pi}\begin{pmatrix} \sin 2\pi x_1 \\ \sin 2\pi x_2 \\ \sin 2\pi(x_1 + x_2) \end{pmatrix}.$$

We have

$$Dw(x) = \begin{pmatrix} \cos 2\pi x_1 & 0 \\ 0 & \cos 2\pi x_2 \\ \cos 2\pi(x_1+x_2) & \cos 2\pi(x_1+x_2) \end{pmatrix}$$

for each $x \in \mathbf{R}^2$. For each $x \in \mathbf{R}^2$ the gradient $Dw(x)$ of w lies in the linear subspace L of $M^{3\times 2}$ given by

$$L = \left\{ \begin{pmatrix} r & 0 \\ 0 & s \\ t & t \end{pmatrix}; r, s, t \in \mathbf{R} \right\}.$$

Let $f: L \to \mathbf{R}$ be defined by

$$f\left(\begin{pmatrix} r & 0 \\ 0 & s \\ t & t \end{pmatrix} \right) = -rst. \tag{3}$$

It is obvious that the only rank-one directions in L are given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Therefore the function f is convex (in fact linear) on each rank-one line contained in L. Using the formula $\cos(a+b) = \cos a \cos b - \sin a \sin b$ we see that

$$\int_{[0,1]^2} f(Dw(x))\,dx = -\int_{[0,1]^2} (\cos 2\pi x_1)^2 (\cos 2\pi x_2)^2\,dx_1\,dx_2 < 0 = f(0). \tag{4}$$

Now we would like to extend f to a rank-one convex function on $M^{3\times 2}$. This does not seem obvious. We can, however, slightly modify f to make the extension easier. In what follows we consider $M^{3\times 2}$ as the six-dimensional euclidean space. The norm of $X \in M^{3\times 2}$ is given by $|X|^2$ = sum of the squares of the entries of X.

LEMMA 2. *Let L be the linear subspace of $M^{3\times 2}$ defined above and let $P: M^{3\times 2} \to L$ be the orthogonal projection. Let $f: L \to \mathbf{R}$ be the function defined by* (3). *Then for each $\varepsilon > 0$ there exists $k = k(\varepsilon) > 0$ such that the function $F: M^{3\times 2} \to \mathbf{R}$ given by*

$$F(X) = f(PX) + \varepsilon |X|^2 + \varepsilon |X|^4 + k |X - PX|^2 \tag{5}$$

is a rank-one convex on $M^{3\times 2}$.

Proof. For every $A, Y \in M^{3\times 2}$ we have

$$\left(\frac{d^2}{dt^2} F(A + tY) \right)_{t=0}$$
$$= \left(\frac{d^2}{dt^2} f(PA + tPY) \right)_{t=0} + 2\varepsilon |Y|^2 + 4\varepsilon |A|^2 |Y|^2 + 8\varepsilon \langle A, Y \rangle^2 + 2k |Y - PY|^2 \tag{6}$$

where $\langle A, Y \rangle = ((d/dt)\,|A + tY|^2/2)_{t=0}$. Since the function $X \to f(PX)$ is a homogenous polynomial of the third degree, there exists $c > 0$ such that

$$\left(\frac{d^2}{dt^2} f(PA + tPY) \right)_{t=0} \geqq -c\,|A|\,|Y|^2$$

for every $A, Y \in M^{3\times 2}$. From (6) we see that

$$\left(\frac{d^2}{dt^2}F(A+tY)\right)_{t=0} \geqq (-c\,|A| + 4\varepsilon\,|A|^2)\,|Y|^2 \tag{7}$$

for every $A, Y \in M^{3\times 2}$. This shows that

$$\left(\frac{d^2}{dt^2}F(A+tY)\right)_{t=0} \geqq 0 \text{ for each } A \in M^{3\times 2} \text{ with } |A| \geqq \frac{c}{4\varepsilon} \text{ and each } Y \in M^{3\times 2}. \quad (*)$$

From (6) we also see that

$$\left(\frac{d^2}{dt^2}F(A+tY)\right)_{t=0} \geqq \left(\frac{d^2}{dt^2}f(PA+tPY)\right)_{t=0} + 2\varepsilon\,|Y|^2 + 2k\,|Y-PY|^2 \tag{8}$$

for every $A, Y \in M^{3\times 2}$. Let us denote the right-hand side of (8) by $g(A, Y, k)$. (Recall that ε is fixed.) Clearly g is continuous on $M^{3\times 2} \times M^{3\times 2} \times \mathbf{R}$. Let

$$\mathcal{K} = \{(A, Y) \in M^{3\times 2} \times M^{3\times 2}, \quad |A| \leqq \frac{c}{4\varepsilon}, \quad \text{rank } Y = 1 \text{ and } |Y| = 1\}.$$

The set $\mathcal{K}$ is clearly a compact subset of $M^{3\times 2} \times M^{3\times 2}$. We claim that there is k_0 such that $g(A, Y, k_0) > \varepsilon$ for all $(A, Y) \in \mathcal{K}$. If not, there would exist for $k = 1, 2, \ldots$ points $(A^{(k)}, Y^{(k)}) \in \mathcal{K}$ such that $g(A^{(k)}, Y^{(k)}, k) \leqq \varepsilon$. We may assume that $(A^{(k)}, Y^{(k)}) \to (\bar{A}, \bar{Y}) \in \mathcal{K}$ as $k \to \infty$ and it follows immediately that $\bar{Y} = P\bar{Y}$ and that $((d^2/dt^2)f(P\bar{A} + tP\bar{Y}))_{t=0} \leqq -\varepsilon$. But this contradicts the fact that f is convex on each rank-one line contained in L. For $k = k_0$ we have

$$\left(\frac{d^2}{dt^2}F(A+tY)\right)_{t=0} > \varepsilon \quad \text{for every } (A, Y) \in \mathcal{K}.$$

Together with $(*)$ this shows that for $k = k_0$ the function F is rank-one convex. The proof is finished. □

Theorem. *There exist $\varepsilon > 0$ and $k > 0$ such that the function F given by* (5) *is rank-one convex but is not quasiconvex.*

Proof. Let w be the function defined above. Since Dw is bounded, we see from (4) that we can choose $\varepsilon > 0$ such that

$$\int_{[0,1]^2} (f(Dw(x)) + \varepsilon\,|Dw(x)|^2 + \varepsilon\,|Dw(x)|^4)\,dx < 0. \tag{10}$$

By Lemma 2 we can find $k = k(\varepsilon)$ such that the function

$$F(X) = f(PX) + \varepsilon\,|X|^2 + \varepsilon\,|X|^4 + k\,|X - PX|^2$$

is rank-one convex. Since $|Dw(x) - PDw(x)| = 0$ for all $x \in \mathbf{R}^2$, we have from (10) that

$$\int_{[0,1]^2} F(Dw(x))\,dx < 0 = F(0) \tag{11}$$

and using Lemma 1 we see that F is not quasiconvex. The proof is finished. □

Remark: Notice that F is a polynomial of degree four.

Corollary. *Let $n \geqq 2$ and $m \geqq 3$ and let $T: M^{m\times n} \to M^{3\times 2}$ can be defined by*

$$TX = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix}.$$

Let $\tilde{F}: M^{m\times n} \to \mathbf{R}$ be defined by $\tilde{F}(X) = F(TX)$ where F is the function from the theorem above. Then $\tilde{F}$ is rank-one convex but is not quasiconvex.

Proof. Since T maps rank-one lines to rank-one lines, $\tilde{F}$ is rank-one convex. Considering the periodic function

$$\tilde{w}(x_1, \ldots, x_n) = (w_1(x_1, x_2), w_2(x_1, x_2), w_3(x_1, x_2), 0, \ldots, 0)$$

where w is the function defined above, we see from (11) that $\tilde{F}$ is not quasiconvex. □

Acknowledgement

I thank Professor J. M. Ball, Professor R. D. James, Professor J. Malý, and Professor L. Tartar for valuable comments.

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(*Issued* 12 *February* 1992)