Introduction to Abstract Algebraic Systems MATH-430-1070 (11365), Fall 2022

MIDTERM 1 SOLUTIONS

Problem 1. (5pts)

(i) State the definition of a group.
(ii) State the definition of a homomorphism.
(iii) State the First Isomorphism Theorem.

A group $(G, \cdot)$ *consists of a set* G *and a function* $\cdot : G \times G \to G$*, such that:*

- *there exists* $e \in G$ *such that:* $e \cdot g = g \cdot e = g$ *for all* $g \in G$,
- *for each* $g \in G$ *there exists* $h \in G$ *such that:* $h \cdot g = g \cdot h = e$*, where* e *is uniquely defined by the previous condition,*
- *for all* $g, h, k \in G$ *there holds:* $(g \cdot h) \cdot k = g \cdot (h \cdot k)$

A homomorphism $\phi : G_1 \to G_2$ *is a function from a group* $(G_1, \cdot)$ *to another group* $(G_2, \cdot)$ *such that:* $\phi(g \cdot h) = \phi(g) \cdot \phi(h)$ *for all* $g, h \in G_1$.

The First Isomorphism Theorem: Let $\phi : G_1 \to G_2$ *be a surjective group homomorphism. Then* G_2 *is isomorphic to the quotient group* $G_1/(Ker\, \phi)$.

Problem 2. (10pts)
Let $(G, \cdot)$ be a finite group whose order is a product of two prime numbers. Let $H_1 \neq H_2$ be two subgroups of G, satisfying $H_1 \neq G$ and $H_2 \neq G$. Prove that $H_1 \cap H_2 = \{e\}$.

We have $|G| = p_1 p_2$ *where* p_1 *and* p_2 *are prime numbers (not necessarily distinct). By Lagrange's theorem and since* $H_1, H_2 \neq G$*, we get:* $|H_1|, |H_2| \in \{1, p_1, p_2\}$. *In particular, both* H_1 *and* H_2 *are cyclic. We argue by contradiction: let* $x \in (H_1 \cap H_2) \setminus \{e\}$*. Then* H_1 *and* H_2 *are generated by* x*, as cyclic groups whose order is a prime number (or 1), which implies* $H_1 = H_2$*. This contradicts the assumption and proves that* $H_1 \cap H_2 = \{e\}$.

Problem 3. (10pts)
Let $(G, \cdot)$ be a group and consider the homomorphism $d : G \to G \times G$ given by $d(g) = (g, g)$. Prove that G is abelian if and only if $Im(d)$ is a normal subgroup of $G \times G$.

If G *is abelian, then so is* $G \times G$ *and consequently its any subgroup (here,* $Im(d)$*) is normal in* $G \times G$*. Vice versa, if* $Im(d)$ *is normal, then:*

$$\forall (g_1, g_2) \in G \times G \quad \forall (g, g) \in Im(d) \qquad (g_1, g_2)(g, g)(g_1, g_2)^{-1} \in Im(d).$$

This means that:

$$\forall g_1, g_2, g \in Im(d) \qquad g_1 g g_1^{-1} = g_2 g g_2^{-1}.$$

Taking $g_2 = e$*, we get:* $g_1 g g_1^{-1} = e$*, i.e.:* $g_1 g = g g_1$*, valid for all* $g, g_1 \in G$*. Hence* G *is abelian.*

Problem 4. (10pts)
Let H_1, H_2 be two subgroups of a group $(G, \cdot)$. Define $\phi : H_1 \times H_2 \to G$ by: $\phi(h_1, h_2) = h_1 h_2$ and assume that ϕ is surjective. Prove that:

(i) ϕ is injective if and only if $H_1 \cap H_2 = \{e\}$,

(ii) ϕ is an isomorphism if and only if $H_1 \cap H_2 = \{e\}$ and both H_1, H_2 are normal subgroups of G.

(i) If ϕ is injective then for every $h \in (H_1 \cap H_2)$ we have: $\phi(h, e)\phi(e, h) = h$, implying that $(h, e) = (e, h)$. This means that $h = e$, proving $H_1 \cap H_2 = \{e\}$. On the other hand, if $H_1 \cap H_2 = \{e\}$, then $\phi(h_1, h_2) = \phi(\bar{h}_1, \bar{h}_2)$ for some $h_1, \bar{h}_1 \in H_1$ and $h_2, \bar{h}_2 \in H_2$ reads: $h_1 h_2 = \bar{h}_1, \bar{h}_2$ or, equivalently: $(\bar{h}_1)^{-1} h_1 = \bar{h}_2 h_2^{-1} \in H_1 \cap H_2$. This yields $(\bar{h}_1)^{-1} h_1 = \bar{h}_2 h_2^{-1} = e$, implying that $h_1 = \bar{h}_1$ and $h_2 = \bar{h}_2$. This ends the proof of injectivity of ϕ.

(ii) If ϕ is an isomorphism then $H_1 \cap H_2 = \{e\}$ by (i). Further, $H_1 \times \{e\}$ and $\{e\} \times H_2$ are normal subgroups of the group $H_1 \times H_2$, because:

$$\forall h_1, \bar{h}_1 \in H_1 \quad \forall h_2, \bar{h}_2 \in H_2 \quad (h_1, h_2)(\bar{h}_1, e)(h_1, h_2)^{-1} = (h_1 \bar{h}_1 h_1^{-1}, e) \in H_1 \times \{e\}$$
$$\text{and } (h_1, h_2)(e, \bar{h}_2)(h_1, h_2)^{-1} = (e, h_2 \bar{h}_2 h_2^{-1}, e) \in \{e\} \times H_2.$$

Since $\phi(H_1 \times \{e\}) = H_1$ and $\phi(\{e\} \times H_2) = H_2$, we conclude that H_1 and H_2 are normal subgroups of $\phi(H_1 \times H_2) = G$.

Conversely, if H_1 and H_2 are normal in G, then:

$$\begin{aligned} \forall h_1, \bar{h}_1 \in H_1 \quad \forall h_2, \bar{h}_2 \in H_2 \quad \phi(((h_1, h_2)(\bar{h}_1, \bar{h}_2)) &= \phi(h_1 \bar{h}_1, h_2 \bar{h}_2) = h_1 \bar{h}_1 h_2 \bar{h}_2 \\ &= h_1 (\bar{h}_1 h_2 (\bar{h}_1)^{-1} (h_2)^{-1}) h_2 \bar{h}_1 \bar{h}_2 = h_1 h_2 \bar{h}_1 \bar{h}_2 \\ &= \phi(h_1, h_2)\phi(\bar{h}_1, \bar{h}_2). \end{aligned}$$

where in the second line above we used that $\bar{h}_1 h_2 (\bar{h}_1)^{-1} (h_2)^{-1} \in H_1 \cap H_2$ in view of normality of H_1 and H_2, resulting in: $\bar{h}_1 h_2 (\bar{h}_1)^{-1} (h_2)^{-1} = e$. Thus ϕ is a homomorphism. It is also surjective by assumption and injective by (i), hence an isomorphism.