Introduction to Abstract Algebraic Systems MATH-430-1070 (11365), Fall 2022

MIDTERM 2

Problem 1. (5pts)
Give the following definitions and statements of theorems:

(i) the Correspondence Theorem for rings,
(ii) the definition of equivalence relation;
(iii) the definition of degree of field extension.

(i) Given a surjective ring homomorphism $\phi : R_1 \to R_2$, there exists a bijection between the set of all ideals in R_2 and the set of ideals in R_1 that contain ker ϕ.
(ii) An equivalence relation in a set S is a binary relation $\sim$ such that for all $a, b, c \in S$ there holds: 1. $a \sim a$ 2. $a \sim b \Rightarrow b \sim a$ 3. $a \sim b \sim c \Rightarrow a \sim c$.
(iii) The degree of the field extension $F \subset K$ is the dimension of the vector space K over the field of scalars F.

Problem 2. (10pts)
Let R be a ring. Prove that $I = \{r \in R;\ \exists k \in \mathbb{N}\ \text{ such that } r^k = 0\}$ is an ideal.

Let $r_1, r_2 \in I$ so that $r_1^{k_1} = 0$ and $r_2^{k_2} = 0$ for some $k_1, k_2 \in \mathbb{N}$. Then:

$$(r_1 - r_2)^{k_1+k_2} = \sum_{i=0}^{k_1+k_2} \binom{k_1 + k_2}{i} r_1^i r_2^{k_1+k_2-i} (-1)^{k_1+k_2-i}.$$

We observe that either $i \geq k_1$ in which case $r_1^{k_1} = 0$, or otherwise $k_1 + k_2 - i > k_2$ in which case $r_2^{k_1+k_2-i} = 0$. Thus each term in the sum above is zero, so $(r_1 - r_2)^{k_1+k_2} = 0$ and thus $r_1 - r_2 \in I$.

Let $r \in I$ so that $r^k = 0$ and let $a \in R$. Then $(ar)^k = a^k r^k = 0$ so $ra \in I$.

Problem 3. (10pts)
Let $F \subset K \subset L$ be finite field extensions. Prove that:

$$[L : F] = [L : K] \cdot [K : F].$$

Let $\{\alpha_i\}_{i=1}^p \subset K$ *be the basis of* K *over* F *and let* $\{\beta_j\}_{j=1}^s \subset L$ *be the basis of* L *over* K. *We first check that* $L = span_K\{\alpha_i\beta_j;\ i = 1 \ldots p, j = 1 \ldots s\}$. *Indeed, for any* $x \in L$ *we have:* $x = \sum_{j=1}^s b_j\beta_j$ *with some* $\{b_j\}_{j=1}^s \subset K$. *For each* b_j, *we also have:* $b_j = \sum_{i=1}^p a_{ji}\alpha_i$ *for some* $\{a_{ji}\}_{j=1}^s \subset F$. *Consequently:*

$$x = \sum_{j=1}^{s}\sum_{i=1}^{p} a_{ji}(\alpha_i\beta_j).$$

We now prove that $\{\alpha_i\beta_j;\ i = 1 \ldots p, j = 1 \ldots s\} \subset L$ *are linearly independent over* K. *If* $\sum_{j=1}^s \sum_{i=1}^p a_{ji}(\alpha_i\beta_j) = 0$ *for some* $\{a_{ji};\ i = 1 \ldots p, j = 1 \ldots s\} \subset K$ *treated as scalars, then there must be* $\sum_{i=1}^p a_{ji}\alpha_i = 0$ *for all* $j = 1 \ldots s$, *because* $\{\beta_j\}_{j=1}^s$ *are linearly independent over* K. *But then each* $a_{ji} = 0$ *because* $\{\alpha_i\}_{i=1}^p$ *are linearly independent over* F.

This proves that $\{\alpha_i\beta_j;\ i = 1 \ldots p, j = 1 \ldots s\} \subset L$ *is a basis of* L *over* K. *Thus:*

$$[L : F] = p \cdot s = [K : F] \cdot [L : K].$$

Problem 4. (10pts)

(a) Given an arbitrary triangle, is it possible to construct a square with the same area?

(b) Given an arbitrary tetrahedron, is it possible to construct a cube with the same volume?

[By "construct" we mean "construct with ruler and compass".]

(a) Given three points in the plane, which are viewed as the vertices of a triangle T, *both the length* a *of a chosen side and the length of the corresponding hight* h *are constructible. This is clear for* a, *while for* h *it follows by the constructability of a line perpendicular to a given line and passing through a given point.*

The area of T *equals* $ah/2$, *so the side of a square* S *with the same area as* T *equals* $b = \sqrt{ah/2}$. *This number is constructible, because products, ratios and square roots of constructible numbers are constructible.*

Finally, constructibility of S *follows from the constructability of segments with with contructible lengths, with one endpoint at a chosen point and perpendicular to other constructible lines.*

(b) A tetrahedron T *with length of one of its sides* a, *length of the planar hight of the base triangle containing the chosen side* h, *and the hight perpendicular to that base* H, *equals* $ahH/6$. *Hence the side of a cube* C *with the same volume as* T, *is* $b = (ahH/h)^{1/3}$. *In general,* b *is algebraic over* $\mathbb{Q}[a, h, H]$ *with degree* 3 *which is not a pure power of* 2. *Hence* b *and so* C *is not constructible.*