Introduction to Abstract Algebraic Systems MATH-430-1070 (11365), Fall 2022

Problem. Show that there exists a maximal ideal in the ring $\mathcal{C}_{(0,1)}$ of continuous real functions on $(0,1)$, which does not have the form:

$$I^{x_0} = \big\{f \in \mathcal{C}_{(0,1)};\ f(x_0) = 0\big\}, \qquad \text{for some } x_0 \in (0,1).$$

Solution. It is easy to check that the following set is an ideal:

$$J = \{f \in \mathcal{C}_{(0,1)};\ f_{|(a,1)} = 0 \text{ for some } a \in (0,1)\}.$$

Since $1 \notin J$, it follows that $J \neq \mathcal{C}_{(0,1)}$ and hence there exists a maximal ideal I in:

$$J \subset I \subset \mathcal{C}_{(0,1)}.$$

This is the ideal with the claimed property that $I \neq I^{x_0}$ for all $x_0 \in (0,1)$. Indeed, given any $x_0 \in (0,1)$, we may find a function $f_{x_0} \in J$ such that $f_{x_0}(x_0) = 1$, so in particular: $f_{x_0} \in J \setminus I^{x_0} \subset I \setminus I^{x_0}$. □