Introduction to Abstract Algebraic Systems MATH-430-1070 (11365), Fall 2022

Exercise Set 5

1. Let G consist of all nonconstant linear maps $\phi : \mathbb{R} \to \mathbb{R}$, i.e.:

$$\phi(x) = ax + b \quad \text{for some } a, b \in \mathbb{R}, \quad a \neq 0.$$

Consider the group $(G, \cdot)$ with the natural multiplicative operation of function composition. Are the following subgroups normal in G:

$$H_1 = \{x \mapsto x + b\}_{b \in \mathbb{R}}, \qquad H_2 = \{x \mapsto ax\}_{a \in \mathbb{R} \setminus \{0\}} \quad ?$$

If they are, what are the quotient groups?

2. Let $H_1 \subset H_2$ be two normal subgroups of a group $(G, \cdot)$. Show that H_2/H_1 is a normal subgroup of G/H_1 and that the groups $(G/H_1)/(H_2/H_1)$ and G/H_2 are isomorphic.

3. Prove that if the group G is not abelian then $G/Z(G)$ cannot be cyclic (recall that the center of G is given by: $Z(G) = \{h \in G;\ hg = gh \quad \text{for all } g \in G\}$).

4. Let H be a subgroup of a group $(G, \cdot)$. Prove that if N is a normal subgroup of G then $N \cap H$ is a normal subgroup of H and the groups $(HN)/N$ and $H/(N \cap H)$ are isomorphic.

5. Let R be a ring and let $f, g \in R[x]$. Assume that the leading coefficient in f is invertible. Prove that there exist unique $q, r \in R[x]$ satisfying:

$$g = fq + r \quad \text{and} \quad \big[\deg(r) < \deg(f) \quad \text{or} \quad r = 0\big]$$