Introduction to Abstract Algebraic Systems MATH-430-1070 (11365), Fall 2022

Exercise Set 6

1. Define the product operation $\circ$ in the set $\mathbb{R}^+$, such that $(\mathbb{R}^+, \cdot, \circ)$ is a ring. (The operation $\cdot$ denotes the usual multiplication in $\mathbb{R}$, here playing the role of addition in the ring).

2. Determine whether the following sets of real-valued functions on $[0, 1]$ are rings, where the operations $+, \cdot$ are taken to be the usual functions' addition and multiplication:
(i) continuous functions,
(ii) polynomial functions whose free term is an integer,
(iii) functions satisfying $f(1/2) = 0$,
(iv) functions satisfying $f(0) = f(1)$,
(v) functions f for whom there exists $k \in \mathbb{Z}$ such that $2^k f(0) = f(1)$.

3. Find all invertible elements in the ring $\mathbb{Z}[\sqrt{-5}]$.

4. Let H_1, H_2 be two normal subgroups of a group $(G, \cdot)$. Show that $H_1 \subset H_2$ if and only if there exists a surjective homomorphism $h : G/H_1 \to G/H_2$ for which the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{id} & G \\ \downarrow & & \downarrow \\ G/H_1 & \xrightarrow{h} & G/H_2 \end{array}$$

Above, the down arrows denote the natural surjective homomorphisms $g \mapsto gH_1$ and $g \mapsto gH_2$.

5. Let R be the ring of all functions $f : \mathbb{R} \to \mathbb{R}$ and let $A \subset \mathbb{R}$.
(i) Show that the set $\{f \in R;\ f_{|A} = 0\}$ is an ideal in R.
(ii) Does every ideal in R have the form indicated in (i)?