Introduction to Abstract Algebraic Systems MATH-430-1070 (11365), Fall 2022

Exercise Set 8

1. Show that every maximal ideal in the ring $\mathcal{C}_{[0,1]}$ of continuous real functions on $[0,1]$, has the form:

$$I^{x_0} = \big\{ f \in \mathcal{C}_{[0,1]};\ f(x_0) = 0 \big\}, \qquad \text{for some } x_0 \in [0,1].$$

2. Show that there exists a maximal ideal in the ring $\mathcal{C}_{(0,1)}$ of continuous real functions on $(0,1)$, which does not have the form:

$$I^{x_0} = \big\{ f \in \mathcal{C}_{(0,1)};\ f(x_0) = 0 \big\}, \qquad \text{for some } x_0 \in (0,1).$$

3. Let $K = \mathbb{Q} \times \mathbb{Q}$. Determine whether $(K, \oplus, \odot)$ is a field, where:

(i) $(a,b) \oplus (c,d) = (a+c, b+d)$ and $(a,b) \odot (c,d) = (ac - bd, ad + bc)$,
(ii) $(a,b) \oplus (c,d) = (a+c, b+d)$ and $(a,b) \odot (c,d) = (ac + 2bd, ad + bc)$.

In each case (i) and (ii) answer the same question if $\mathbb{Q}$ is replaced by $\mathbb{R}$.

4. Let $a, b \in \mathbb{Q} \setminus \{0\}$. Prove that the fields $\mathbb{Q}(\sqrt{a})$ and $\mathbb{Q}(\sqrt{b})$are isomorphic if and only if $\frac{a}{b}$ is a square of a rational number.

5. Define $R = \{z \in \mathbb{C};\ z^{(2^n)} = 1 \text{ for some } n \in \mathbb{N}\}$ and consider the ring $(R, +, \cdot)$ where $+$ is the multiplication of complex numbers and $\cdot$ is the trivial product, namely: $r_1 \cdot r_2 = 0$ for all $r_1, r_2 \in R$. Then R is a ring without unity element. Prove that R has no maximal ideal.