Introduction to Abstract Algebraic Systems MATH-430-1070 (11365), Fall 2022

Exercise Set 9

1. Which of the following numbers are algebraic over $\mathbb{Q}$? In each positive case, determine the degree of the algebraic number.

(i) $1+\sqrt{2}+\sqrt{3}$,
(ii) $\sqrt[6]{3}+\sqrt{3}$,
(iii) $1+\sqrt{\pi}$,
(iv) $1+\sqrt{2}+\sqrt{4}+\sqrt{8}+\ldots+\sqrt{2^{n-1}}$,
(v) $\sqrt[4]{5}+\sqrt{5}$,
(vi) $2+\sqrt[3]{2}+\sqrt[3]{4}$.

2. Let $r \in \mathbb{Z}$ be such that it is not divisible by the square of any prime number. Prove that $K = \mathbb{Q} \times \mathbb{Q}$ with the following operations is a field:

$$(a,b) \oplus (c,d) = (a+c, b+d) \qquad \text{and} \qquad (a,b) \odot (c,d) = (ac+rbd, ad+bc).$$

Prove that K is in fact isomorphic with $\mathbb{Q}(\sqrt{r})$.

3. Prove the following:

(a) a complex number z is algebraic over $\mathbb{Q}$ if and only if $\bar{z}$ is algebraic over $\mathbb{Q}$,

(b) two real numbers a, b are algebraic over $\mathbb{Q}$ if an only if the complex number $a+bi$ is algebraic over $\mathbb{Q}$.

4. Give an example of a field extension $K \subset L$ and two elements $a, b \in L$ that are algebraic over K with unequal degrees, and such that:

$$[K(a,b) : K] < (\deg_K a) \cdot (\deg_K b).$$

5. Let $K \subset L$ be a field extension and let $a \in L$ be algebraic over K. Prove that if $\deg_K a$ is odd then $K(a) = K(a^2)$. Is the converse implication true?