Problems in Pattern Formation, Geometry and Design of Materials

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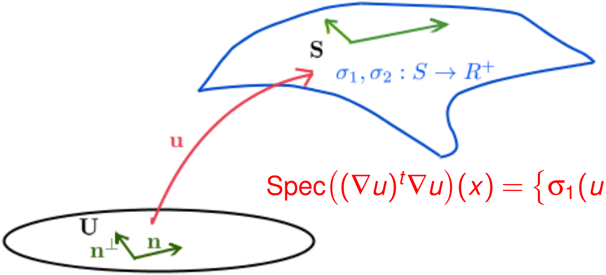
Pattern formation, geometry and design of materials

- Manipulating the micro-structure of material $\Longrightarrow$ desired change in its mechanical responses.
  - Actuation of thin nematic glass sheets
  - Halftone gel lithography
  - Understanding morphogenesis in response to inhomogeneous and incompatible prestrain
  - Pattern formation: wrinkling / blistering / crumpling ...
- Mathematical problems combining geometry, analysis, calculus of variations, pdes, related to practical questions of material design.
  - Forward and inverse problems of isometric immersions
  - Energy scaling laws
  - Incompatible elasticity / dimension reduction
  - Questions of regularity / uniqueness / existence of solutions to nonconvex problems

Geometric design problem in actuation of liquid crystals



Given: S, σ_1, σ_2
Find: U, u

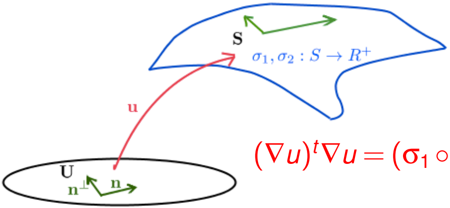
so that:

$$\mathrm{Spec}\big((\nabla u)^t \nabla u\big)(x) = \big\{\sigma_1(u(x)), \sigma_2(u(x))\big\}$$

- Find eigenvectors $\vec{n}(x), \vec{n}^\perp(x)$ of $(\nabla u)^t \nabla u$
- Allign the liquid crystal director in the flat configuration with $\vec{n}(x)$
- Spontaneous actuation of the metric G will result in manufacturing surface S:

$$(\nabla u)^t \nabla u = (\sigma_1 \circ u)\vec{n} \otimes \vec{n} + (\sigma_2 \circ u)\vec{n}^\perp \otimes \vec{n}^\perp = G$$

Geometric design problem in actuation of liquid crystals

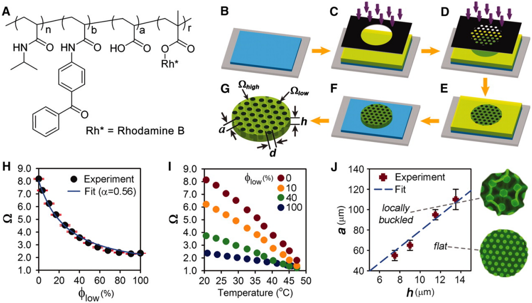


May not yield S !

$$(\nabla u)^t \nabla u = (\sigma_1 \circ u)\vec{n} \otimes \vec{n} + (\sigma_2 \circ u)\vec{n}^\perp \otimes \vec{n}^\perp = G$$

- Among $y : U \to \mathbb{R}^3$ satisfying $(\nabla y)^t \nabla y = G$, the energy minimizing y will be realised: $E(y) = \int_U |H_y|^2 \equiv \int_U |II_y|^2$
- Another problem (more restrictive):
  Given $g : S \to \mathbb{R}^{2\times 2}_{sym,pos}$ prestrain metric on S
  Find $u : U \to S$ such that $(\nabla u)^t \nabla u = g \circ u = G$
- Energy minimization criterion as before. [Acharya-L-Pakzad'14]

Half-tone gel lithography



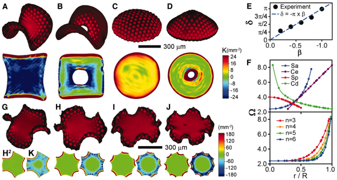
[Kim, Hanna, Byun, Santangelo, Hayward – Science, 2012]

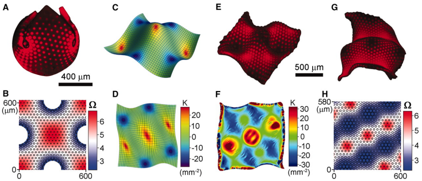
- Method of photopatterning polymer films that yields temperature-responsive gel sheets that can transform between a flat state and a prescribed 3d shape
- Lightly cross-linked dots embedded in a cross-linked matrix $\Longrightarrow$ "nearly continuous" 2d "patterns of swelling" (prestrain metric G)

Half-tone gel lithography

- Forward problem:
  Given G on U, minimize: $E^h(y) = h^2 \int_U |II_y|^2 + \int_U |I_y - G|^2$
- Euler-Lagrange equations given in terms of $I = I_y$ and $II = II_y$:

$$(*) \quad \begin{cases} F(I, II; G) = 0 \\ \text{compatibility conditions via Gauss-Codazzi eqns} \\ \text{boundary conditions} \end{cases}$$





[Kim, Hanna, Byun, Santangelo, Hayward – Science, 2012]

- Inverse design problem: [Dias, Hanna, Santangelo – Phys. Rev E, 2011]
  Given S with parametrisation $u : U \to S$, yielding: $I = I_u, \quad II = II_u$
  Solve $(*)$ for $G \implies$ imprint $G \implies$ activate for S

General set-up: Incompatible elasticity

$$E(u) = \int_\Omega W(\nabla u \sqrt{G}^{-1})\, dx, \qquad u : \mathbb{R}^3 \supset \Omega \to \mathbb{R}^3$$

- $E(u) = 0$ if and only if: $(\nabla u)^t \nabla u = G$ and $\det \nabla u > 0$
- $Riem(G) \neq 0 \implies \inf_{u \in W^{1,2}(\Omega,\mathbb{R}^3)} E(u) > 0$

- Dimension reduction for prestrained thin films:

$$\Omega^h = U \times (-\tfrac{h}{2}, \tfrac{h}{2}), \quad E^h(u^h) = \tfrac{1}{h} \int_{\Omega^h} (\nabla u \sqrt{G}^{-1}), \quad G(x', x_3) = G(x')$$

Questions: 1. Scaling: $\inf E^h \sim h^\beta$, 2. Asymptotics: $\operatorname{argmin} E^h$ as $h \to 0$

- Small energy theories $\beta \geq 2 \implies$ only 2 residual theories!

Theorem (L, Pakzad'09; Bhattacharya, L, Schaffner'14)

$\beta = 2$. *Only valid when:* $\exists y \in W^{2,2}(U, \mathbb{R}^3)\ \ (\nabla y)^t \nabla y = G_{tan}$.
Then: $argmin E^h \to argmin\, \mathcal{I}_2(y) = \int_U |sym((\nabla y)^t \nabla \vec{b})|^2$.

Theorem (L, Raoult, Ricciotti'15)

$\beta = 4$. *Only valid when:* $R_{1212} = R_{1213} = R_{1223}(G) = 0$.
$argmin E^h \to argmin\, \mathcal{I}_4 = \int_U |stretching|^2 + |bending|^2 + \int_U |Riem(G)|^2$.

General set-up: Incompatible elasticity

- $\beta = 2$. Only valid when: $\exists y \in W^{2,2}(U, \mathbb{R}^3)$ $(\nabla y)^t \nabla y = G_{tan}$.
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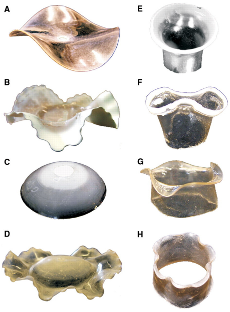
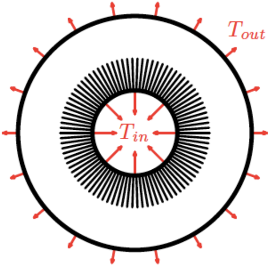
- $\beta = 4$. Only valid when: $R_{1212} = R_{1213} = R_{1223}(G) = 0$.
  $\mathcal{I}_4 = \int_U |\mathrm{stretching}|^2 + |\mathrm{bending}|^2 + \int_U |\mathrm{Riem}(G)|^2$.
- Questions:



[Klein, Efrati, Sharon - Science '07]

- Uniqueness / multiplicity of minimizers to $\mathcal{I}_2$
- Uniqueness /multiplicity of minimizers to the "linearized problem":
  $\mathcal{I}_{2,lin}(v) = \int_U |\nabla^2 v|^2$; $\det \nabla^2 v = \mathrm{curl}^t \mathrm{curl}\ G_{tan}$
- Differences for positive / negative curvature of G_{tan}
- Geometry enters in a subtle manner; regularity questions for isometric immersions

- In lower energy regimes, the minimizing sequences develop oscillatory behaviour $\Longrightarrow$ energy relaxation
- Energy comparison methods $\Longrightarrow$ certain patterns are energetically preferable
- Example: Wrinkles of annular sheet loaded in the radial direction [Kohn, Bella – CPAM, 2014]



$$E^h(u^h) = \frac{1}{h}\int_{\Omega^h} W(\nabla u^h) + \left(\frac{T_{in}}{h}\int_{|x|=r} u\cdot\frac{x}{r} - \frac{T_{out}}{h}\int_{|x|=T} u\cdot\frac{x}{R}\right)$$

Then: $|\min_{u^h\in W^{1,2}} E^h - \min E_0| = O(h)$

where E_0 = relaxation obtained by quasi-convexification + boundary terms

Wrinkling/ blistering/ crumpling ...

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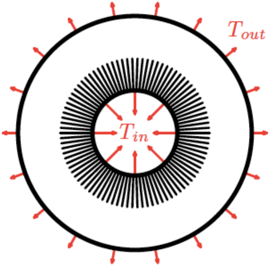

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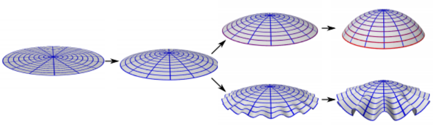
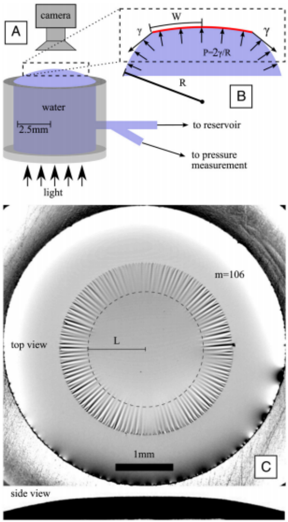
where E_0 = relaxation obtained by quasi-convexification + boundary terms

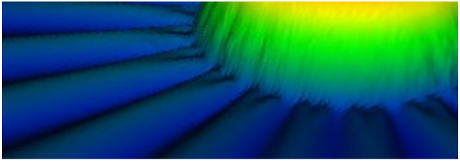
- Lower bound estimates: $\min E^h - \min E_0 \geq ch$
- Upper bound construction of u^h, informed by experimental ansatz



Wrinkling/ blistering/ crumpling ...

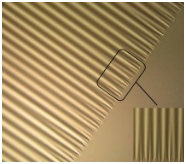


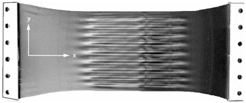
[King, Schroll, Davidovitch, Menon - PNAS’12]



[Davidovitch et all, 2012]

- Wrinkle to crumple transition
- Smooth cascades
- Other patterns





Cerda, Mahadevan – PRL’03

[Huang, Davidovitch, Santangelo, Russell, Menon – PRL’10]

Rigidity and flexibility

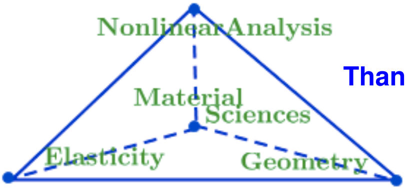
- Oscillatory behaviour $\Longrightarrow$ lower regularity in the limit $\Longrightarrow$ analytical difficulties (low regularity maps are "flexible")

Theorem (Nash, Kuiper – 1955. Convex integration for isom. immer.)

Let G be a $\mathcal{C}^2$ metric. Then its $\mathcal{C}^1$ isometric immersions are dense (w.r.t $\mathcal{C}^0$ norm) in the set of all short immersions of G.

Theorem (L-Pakzad – 2015. Convex integration for Monge-Ampère)

Let f be Hölder continuous. Then $\mathcal{C}^{1,\alpha}$ solutions (for any $\alpha < \frac{1}{7}$) to $\det \nabla^2 v = f$ are dense in $\mathcal{C}^0(U)$.



Thank you for your attention