Local behaviour of solutions to nonstandard growth measure data problems

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Goals

$$-\mathrm{div}\mathcal{A}(x, Du) = \mu \quad \text{in} \quad \Omega \subset \mathbb{R}^N$$

with nonnegative bounded measure μ and Carathéodory's function $\mathcal{A} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N \implies$ nonlinear operator (including Δ and Δ_p).

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Problems:

- definition of solution
- Orlicz growth (no homogeneity $\mathcal{A}(x, k\xi) = |k|^{p-2}k\mathcal{A}(x, \xi)$)
- measurable dependence $x \mapsto \mathcal{A}(x, \xi)$

Measure data problems

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$$-\mathrm{div}\mathcal{A}(x, Du) = \mu,$$

where $\mathcal{A}(x,\xi)\cdot\xi \simeq G(|\xi|) \Leftarrow$ doubling e.g. $G_{p,\alpha}(s) = s^p \log^\alpha(1+s)$

Who can be called 'a solution'?

A function $u \in W^{1,G}_{loc}(\Omega)$ is called a weak solution to a problem

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, Du) = \mu \quad \text{in} \;\; \Omega, \\ u = 0 \quad \text{on} \;\; \partial\Omega, \end{cases}$$

$$\text{if} \quad \int_\Omega \mathcal{A}(x, Du) \cdot D\phi \, dx = \int_\Omega \phi \, d\mu(x) \quad \text{for every} \;\; \phi \in C_c^\infty(\Omega).$$

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Weak solutions are too restrictive,
distributional solutions can be wild... :(

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Wild, but not too wild

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Well, already for $-\Delta_p u = \delta_0$ in $B(0,1)$
we deal with the so-called fundamental solution

$$G(x) = c_{n,p} \begin{cases} \left(|x|^{\frac{p-n}{p-1}} - 1\right) & \text{if } 1 < p < n, \\ -\log|x| & \text{if } p = n, \end{cases} \qquad |x| \neq 0,$$

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One may study various kids of very weak solutions:
SOLA (Boccardo&Gallouët ’89), renormalized solutions (DiPerna&Lions ’89, Boccardo, Giachetti, Diaz, Murat ’93), entropy solution (Bénilan, Boccardo, Gallouët, Gariepy, Pierre, Vazquez, Murat ’95), or (Kilpeläinen, Kuusi, Tuhola-Kujanpää ’11)

$\mathcal{A}$-superharmonic functions.

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A <u>continuous</u> function $u \in W^{1,G}_{loc}(\Omega)$ is an $\mathcal{A}$-*harmonic* function in an open set Ω if it is a (weak) solution to $-\mathrm{div}\mathcal{A}(x, Du) = 0$.

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$\mathcal{A}$-super/subharmonicity

We say that a lower semicontinuous function u is $\mathcal{A}$-superharmonic if for any $K \Subset \Omega$ and any $\mathcal{A}$-harmonic $h \in C(\overline{K})$ in K, $u \geqslant h$ on ∂K implies $u \geqslant h$ in K (u is $\mathcal{A}$-subharmonic if $(-u)$ is $\mathcal{A}$-superharmonic).

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This guy we want to 'control by a potential' and prove its regularity.

Potential estimate in the linear case 1/2

Global case

If u solves $-\Delta u = \mu$ in $\mathbb{R}^N$, then

$$u(x) = \int_{\mathbb{R}^N} G(x,y)\, d\mu(y)$$

with Green's function

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Potential estimate in the linear case 2/2

Local behaviour of solutions to $-\Delta u = \mu$

Localized/trucated Riesz potential of a nonnegative measure

$$\mathrm{I}_2^{\mu}(x,R) := \int_0^R \frac{|\mu|(B_\varrho(x))}{\varrho^{n-2}}\frac{d\varrho}{\varrho} \lesssim_n \int_{B_R(x)} \frac{d|\mu|(y)}{|x-y|^{n-2}}$$

$$\leqslant \int_{\mathbb{R}^N} \frac{d|\mu|(y)}{|x-y|^{n-2}} = \mathrm{I}_2(|\mu|)(x) \quad \Leftarrow \text{Riesz potential}$$

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$$|u(x)| \leqslant C\left(\mathrm{I}_2^{\mu}(x,R) + \text{`}\textit{sth not that much important}\text{'}\right).$$

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$-\Delta_p u = -\mathrm{div}(|Du|^{p-2}Du) = \mu$ **for** $1 < p < \infty$

Expecting

$$|u(x)| \leqslant C \left(\mathcal{W}_p^{\mu}(x,R) + \text{`}sth(u,R) \text{ not that much important}' \right),$$

we have to employ another potential

$$\mathcal{W}_p^{\mu}(x,R) = \int_0^R \left(\frac{|\mu|(B_\varrho(x))}{\varrho^{n-1}} \right)^{\frac{1}{p-1}} d\varrho$$

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Kilpeläinen & Malý ['92,'94] proven that for $\mu \geqslant 0$ we actually have

$$\mathcal{W}_p^\mu(x,R) \lesssim u(x) \lesssim \mathcal{W}_p^\mu(x,2R) + \text{`}sth(u,R)'$$

Trudinger & Wang [2002], Korte & Kuusi [2010], Kuusi & Mingione [2018]

The operator of general growth

Growth & ellipticity condition

$$c_1^{\mathcal{A}} G(|\xi|) \leqslant \mathcal{A}(x,\xi) \cdot \xi \qquad \text{and} \qquad |\mathcal{A}(x,\xi)| \leqslant c_2^{\mathcal{A}} g(|\xi|),$$

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$$-\operatorname{div}\left(a(x)\frac{G(|Du|)}{|Du|^2}Du\right) = \mu \quad \text{with} \quad 0 \ll a \in L^\infty(\Omega)$$

Theorem

Assume that u is a nonnegative function being $\mathcal{A}$-superharmonic and finite a.e. in $B(x_0, R_{\mathcal{W}}) \Subset \Omega$ for some $R_{\mathcal{W}}$. Let (Havin-Mazy'a-)Wolff potential be given by

$$\mathcal{W}_G^{\mu_u}(x_0, R) = \int_0^R g^{-1}\left(\frac{\mu_u(B(x_0, r))}{r^{n-1}}\right) dr$$

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$$C_L\left(\mathcal{W}_G^{\mu_u}(x_0, R) - R\right) \leqslant u(x_0) \leqslant C_U\left(\inf_{B(x_0,R)} u(x) + \mathcal{W}_G^{\mu_u}(x_0, R) + R\right)$$

with $C_L, C_U > 0$ depending only on parameters $i_G, s_G, c_1^{\mathcal{A}}, c_2^{\mathcal{A}}, n$.

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* Similar upper bound was proven by Malý [2003] for $\mathcal{A}$-superminimizer.

Powerful corollaries

$u \geqslant 0$ **is** $\mathcal{A}$**-superharmonic and finite a.e. and** $\mu_u := -\mathrm{div}\mathcal{A}(x, Du)$ **(distrib.)**

- The result is sharp as the same potential controls bounds from above and from below.
- u is continuous in $x_0 \iff \mathcal{W}_G^{\mu_u}(x,r)$ is small for $x \in B(x_0, r)$.
- if $-\mathrm{div}\mathcal{A}(x, Du) = \mu_u = \delta_{x_0}$; x is close to x_0, $r = |x - x_0|$, then

$$c^{-1}\left(\int_r^{2r} g^{-1}\left(s^{1-n}\right)\, ds - r\right) \leqslant u(x) \leqslant c\left(\int_r^{2r} g^{-1}\left(s^{1-n}\right)\, ds + \inf_{B_{2r}} u + r\right).$$

If additionally G is so fast in infinity that $\int_0 g^{-1}\left(s^{1-n}\right)\, ds < \infty$, then $u \in L^\infty(B_r)$. This bound is optimal.

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$u \geqslant 0$ is $\mathcal{A}$-superharmonic and finite a.e. and $\mu_u := -\mathrm{div}\mathcal{A}(x, Du)$ (distrib.)

- $u \in C^{0,\beta}_{loc}(\Omega) \iff \mu_{u,\theta}(B(x,r)) \leqslant c r^{n-1} g(r^{\theta-1}) \simeq r^{n-\theta} G(r^{\theta-1})$ (Orlicz-Morrey-type condition; * [C., Karppinen, 2019])

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- For more see **C., Giannetti, Zatorska–Goldstein**, Wolff potentials and local behaviour of solutions to measure data elliptic problems with Orlicz growth, arXiv:2006.02172

Thank you for your attention!

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